SAGINAW VALLEY STATE UNIVERSITY
SOLUTIONS OF 2013 MATH OLYMPICS LEVEL I

1. $\sqrt{\frac{1}{9}+\frac{1}{16}} = ?$
(a) $\frac{7}{12}$ **(b)** $\frac{1}{5}$ **(c)** $\frac{2}{7}$ **(d)** $\frac{5}{12}$ **(e)** none of the above

The answer is: **(d)**
Solution:

$$\sqrt{\frac{1}{9}+\frac{1}{16}} = \sqrt{\frac{9+16}{9\cdot 16}} = \frac{5}{3\cdot 4} = \frac{5}{12}.$$

2. Suppose the operation $*$ is defined on the set of integers by $a * b = a + 2b$. Then for every two integers a and b, the value of $a * (b * a)$ is the same as
(a) $(5a) * b$ **(b)** $a * b$ **(c)** $b * a$ **(d)** $b * (4a)$
(e) none of the above

The answer is: **(a)**
Solution: Using the definition of $*$, we have
$a * (b * a) = a * (b + 2a) = a + 2(b + 2a) = 5a + 2b = (5a) * b.$

3. Given that the vertex of the parabola $y = x^2 + 8x + k$ is on the x-axis, what is the value of k ?
(a) 4 **(b)** 24 **(c)** 16 **(d)** 0 **(e)** 8

The answer is: **(c)**
Solution: Since, in completing the square, $x^2 + 8x + k = (x + 4)^2 + k - 16$, the equation of the parabola can be written as $y = (x + 4)^2 + k - 16$. The vertex is $(-4, k - 16)$ which is on the x-axis when $k = 16$.

4. John and Nancy live on the same street and often walk towards each other's home. If they both leave their homes at 8:00 a.m., then they will meet at 8:04 a.m. If Nancy leaves her home at 8:00 a.m. but John does not leave his home until 8:03 a.m., then they will meet at 8:05 a.m. How many minutes does it take for John to walk all the way to Nancy's home? Assume that each person walks at his or her own constant rate.

(a) 8 minutes **(b)** 9 minutes **(c)** 10 minutes **(d)** 11 minutes
(e) none of the above

The answer is: **(e)**

Solution: Let x be John's walking speed and y be Nancy's walking speed both in feet per minute, and let d denote the distance in feet between the houses. Then in 4 minutes, John travels $4x$ feet and Nancy travels $4y$ feet so that (from the information given) $d = 4x + 4y$. Similarly, one gets $d = 2x + 5y$. It follows that $d = 5d - 4d = 5(4x + 4y) - 4(2x + 5y) = 12x$. This implies it will take John 12 minutes to travel distance d.

5. Which of these numbers is the largest?

(a) $\sqrt{\sqrt[3]{5 \cdot 6}}$ **(b)** $\sqrt{6\sqrt[3]{5}}$ **(c)** $\sqrt{5\sqrt[3]{6}}$ **(d)** $\sqrt[3]{5\sqrt{6}}$ **(e)** $\sqrt[3]{6\sqrt{5}}$

The answer is: **(b)**

Solution: Note that if a and b are positive, then $a < b$ if and only if $a^6 < b^6$. Since all the choices are positive, raise each to the sixth power to simplify the comparison. We then have in each case:
(a) $5 \cdot 6$ (b) $6 \cdot 6 \cdot 6 \cdot 5$ (c) $5 \cdot 5 \cdot 5 \cdot 6$ (d) $5 \cdot 5 \cdot 6$ (e) $6 \cdot 6 \cdot 5$.
Thus the largest is (b).

6. The graph of the equation $x^2 - xy + x - y = 0$ is

(a) a parabola **(b)** a point **(c)** an ellipse **(d)** a line
(e) a pair of intersecting lines

The answer is: **(e)**

Solution: Factoring the left-hand side of the equation $x^2 - xy + x - y = 0$, we obtain $(x + 1)(x - y) = 0$. The latter is satisfied by all points (x, y) satisfying at least one of $x+1 = 0$ and $x-y = 0$. In other words, the graph of $x^2-xy+x-y = 0$ is composed of the two lines determined by the equations $x + 1 = 0$ and $x - y = 0$ (intersecting at $(-1, -1)$).

7. In a class of 100 students, there are 50 who play soccer, 45 who play basketball, and 50 who play volleyball. Only 15 of these students play all three sports. Everyone plays at least one of these sports. How many of the students play exactly two of these sports?

(a) 20 **(b)** 35 **(c)** 25 **(d)** 15 **(e)** none of the above

The answer is: **(d)**

Solution: Let S_1 be the number of students who play exactly one of the three sports, S_2 be the number of students who play exactly two of the three sports, and let S_3 be the number of students who play all three sports. We know that $S_3 = 15$ and $S_1+S_2+S_3 = 100$ (the total number of students). Also in the sum 50+45+50 (soccer players + basketball players + volleyball players), each student who plays exactly one sport is counted once, the ones who play exactly two sports are counted twice, and those who play all three sports are counted three times. Thus, $50 + 45 + 50 = S_1 + 2S_2 + 3S_3$. Note that $S_2 = (S_1 + 2S_2 + 3S_3) - (S_1 + S_2 + S_3) - 2S_3 = 15$.

8. What is the smallest number of seats in a large auditorium that must be occupied in order to be certain that at least two people share the same first and last initials?

(a) 675 **(b)** 677 **(c)** 51 **(d)** 53 **(e)** none of the above

The answer is: **(b)**

Solution: We shall use a "pigeon hole principle" argument. We are looking for identical first and last initials. There are 26 letters in the alphabet, so there are 26^2 different groups of people distinguished by their first and last initial. Thus, if we have $26^2 + 1 = 677$ occupied chairs there will be at least two people who will be in the same group based on their initials. Any number of chairs less than $26^2 + 1$ allows for a situation when all people will have different first and last initial since there are 26^2 different groups of people.

9. The number of real solutions of the equation $\mid x - 2 \mid + \mid x - 3 \mid = 1$ is

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** none of the above

The answer is: **(e)**

Solution: For any x in the interval $[2, 3]$, we do have
$\mid x-2 \mid + \mid x-3 \mid = (x-2)+(3-x) = 1$. So there are infinitely many real solutions.

10. Suppose that only eight tiles are left in the scrabble bag and the letters on the tiles spell CALCULUS. How many ways (if the order of choosing doesn't matter) can you choose two tiles?

(a) 23 **(b)** 13 **(c)** 20 **(d)** 10 **(e)** none of the above

The answer is: **(b)**

Solution: There are five different letters in the bag A, C, L, S, and U. There are three ways to select the double letters CC, LL and UU. The remaining selections use two distinct letters. Since there are five letters we have $\left(\begin{array}{c} 5 \\ 2 \end{array} \right) = 10$ ways of choosing two distinct letters. Hence, there are 13 ways to choose two tiles if order of choosing doesn't matter.

Note: If order does matter, for example, if drawing AC is treated as different from CA, then there are $5 \quad 4 = 20$ different ordered pairs. Hence, there are 23 ways to choose two tiles if order is taken into account.

11. Yannick has a total of \$200 in his two pockets. He takes one fourth of the money in his left pocket and puts it in his right pocket. He then takes \$20 from his left pocket and puts it in his right pocket. If he now has an equal amount of money in each pocket, then how much money did he originally have in his left pocket?

(a) \$120 **(b)**\$160 **(c)** \$180 **(d)** \$140 **(e)** \$80

The answer is: **(b)**

Solution: Working the problem backward we observe that: Yannick had \$100 in his left and \$100 in his right pocket at the end. So, he had \$120 in his left pocket (and \$80 in his right pocket) before moving \$20 from his left to his right pocket. Thus, solving $x - \frac{x}{4} = 120$, where x represents the original amount he has, to see that Yannick had \$160 in his left pocket originally.

Alternatively, let R be the amount of dollars in the right pocket and L the amount in the left pocket, then $R + L = 200$ and $R + (1/4)L + 20 = (3/4)L - 20$. Then replace R in the later equation by $200 - L$ and solve for L to get $L = 160$.

12. A boat is traveling against the flow of a river. Suppose the river is flowing at a constant speed and the boat maintains a constant speed with respect to the river while traveling in either direction along the river, in other words, the boat would maintain a constant speed in still water. At a certain moment of time a blow-up ball falls off the boat and starts floating down the river. 20 minutes after the ball fell into the water this was noticed and the boat reversed its direction and started going down the river chasing the ball. How long was the ball in the water before it was retrieved?

(a) 20 minutes **(b)** 10 minutes **(c)** 60 minutes
(d) 40 minutes **(e)** none of the above

The answer is: **(d)**

Solution 1: Let T be the time after which it was noticed that the ball is missing. In the given set-up $T = 20$. However, let us consider a different set-up, based on the given one, where we reduce the speed of the river and that of the boat by the same quantity. We shall compute the time the ball was in the water in this more convenient set-up, but the answer will be enough to find the answer of the question in the given problem. Notice that everything depends on $v - c$ and $v + c$, where c is the speed of the river relative to the banks of the river and v is the speed of the the boat relative to the river. Thus, $v - c$ and $v + c$ are the speed of the boat relative to the banks of the river when going up-stream and down-stream, respectively. Notice that the differences $v - c$ and $v + c$ do not change if we subtract or add the same quantity from both v and c. So, let us reduce all velocities by the velocity of the river, i.e, in the new "world" the river will become stationary, which means that the boat will be traveling with the same speed going either up or down the river. In particular, if it takes T minutes to notice the missing ball, it will take another T minutes to get back to it (the ball is not moving!), hence the ball will be overall $2T$ minutes in the water. Now, if we go back to the given world we only need to notice that relative quantities will be preserved, hence it will be again $2T$ minutes until the ball is retrieved.

Solution 2: Alternatively, let T_1 be the time the boat travels back down the river chasing the ball. So the ball is in the water $T + T_1$ minutes. During this time the ball traveled the distance $(T + T_1)c$, while the boat traveled a distance $(v - c)T$ up the river and $(v + c)T_1$ down the river (all distances are measured with respect to the banks of the river). Since at the end, the boat and the ball are at the same place, from the place the ball was dropped, we have $(T+T_1)c = T_1(v+c) - T(v-c)$. Simplifying we find $0 = T_1 v - T v$, hence $T_1 = T = 20$ and it will be again 40 minutes until the ball is retrieved.

13. It takes 3 hours filling a pool using two pipes. It takes 5 hours to fill the pool using only the larger pipe. How long does it take to fill the pool using only the smaller pipe?

(a) 1.5 hours **(b)** 4 hours **(c)** 7.5 hours **(d)** 5 hours
(e) none of the above

The answer is: **(c)**

Solution: Let x be the amount of time it takes for the smaller pipe to fill the pool. Assuming that the smaller and larger pipe are filling at a constant rate, then the smaller pipe fills the pool at the rate of $1/x$, while the larger pipe fills the pool at the rate of $1/5$. We know that if both are filling the pool it will take 3 hours, i.e., $\frac{1}{x} + \frac{1}{5} = \frac{1}{3}$.
Multiplying both sides by $15x$ gives $15 + 3x = 5x$, i.e., $2x = 15$ and $x = 7.5$ hours.

14. In a certain football league, the only way to score is to kick a field goal for 3 points or score a touchdown for 7 points. Thus the scores such as 1, 4 and 8 are not possible. How many positive scores are not possible?
(a) 5 **(b)** 6 **(c)** 9 **(d)** 11 **(e)** infinitely many

The answer is: **(b)**
Solution: One checks directly that the following list of scores up to 14 is the complete list of obtainable scores up to that point: 3, 6, 7, 9, 10, 12, 13, 14. Now, we have 3 consecutive scores, namely 12, 13, and 14, which are obtainable, and this implies every score greater than 14 is obtainable. To see this, observe that if n is an obtainable score, then so is $n + 3$ (simply add another field goal to whatever it took to get n points); hence, $15 = 12+3$, $16 = 13+3$, and $17 = 14+3$ are all obtainable and so are $18 = 15+3$, $19 = 16+3$, $20 = 17+3$, and so on. Therefore, the positive integral scores which are not obtainable are 1, 2, 4, 5, 8, and 11. Thus, the answer is 6.

15. For all x such that $x \neq 2, \frac{-2}{5}, 0$, what is $\frac{-24}{4x+8x^2-5x^3} + \frac{1}{2-x}$ equal to?
(a) $\frac{12}{5x^2+2x}$ **(b)** $\frac{5x^2+12x}{2x^2+5x^3}$ **(c)** $\frac{-5}{5x^2+2x}$ **(d)** $\frac{-5-\frac{12}{x}}{5x+2}$
(e) none of the above

The answer is: **(d)**
Solution:

$$\begin{aligned}\frac{-24}{4x+8x^2-5x^3}+\frac{1}{2-x} &= \frac{-24}{x(2-x)(2+5x)}+\frac{1}{2-x}\\ &= \frac{-24}{x(2-x)(2+5x)}+\frac{x(2+5x)}{x(2+5x)(2-x)}\\ &= \frac{-24+2x+5x^2}{x(2+5x)(2-x)}\\ &= \frac{(x-2)(5x+12)}{x(2+5x)(2-x)}\end{aligned}$$

Now when $x \neq 2, \frac{-2}{5}, 0$, we do have

$$\frac{-24}{4x-x^2-5x^3}+\frac{1}{2-x}=\frac{-(2-x)(5x+12)}{x(2+5x)(2-x)}=\frac{-5x-12}{5x^2+2x}$$

So it does not satisfy (a) and (c). The next line below shows that it also does not satisfy (b):

$$\frac{5x^2+12x}{2x^2+5x^3}=\frac{x(5x+12)}{x(2x+5x^2)}=\frac{5x+12}{5x^2+2x}.$$

However,

$$\frac{-5-\frac{12}{x}}{5x+2}=\frac{(-5-\frac{12}{x})\cdot x}{(5x+2)\cdot x}=\frac{-5x-12}{5x^2+2x}$$

Hence it does satisfy (d).

16. For how many integers n between 1 and 100 does x^2+x-n factor into the product of two linear factors with integer coefficients?

(a) 9 **(b)** 2 **(c)** 1 **(d)** 10 **(e)** none of the above

The answer is: **(a)**

Solution 1: Since $-n<0$, it follows that $x^2+x-n=(x-a)(x+b)$ where a and b are positive integers. Thus $n=ab$ and $b=1+a$ by expanding the right hand side of the above equality. Here are the possible choices for a and b:

a: 1 — 2 —. . .— 9 — 10

b: 2 — 3 —. . .— 10 — 11

$n=ab$: $1\cdot 2=2$ — $2\cdot 3=6$— . . . —$9\cdot 10=90$ — $10\cdot 11=110$.

There are 9 values for n between 1 and 100.

Solution 2: By the quadratic formula, $x^2+x-n=(x-x_1)(x-x_2)$ where

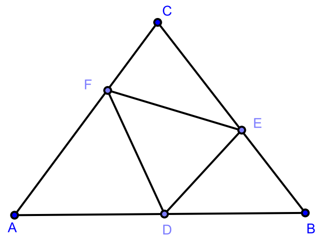
$$x_1, x_2=\frac{-1\pm\sqrt{1+4n}}{2}.$$

Since x_1 and x_2 are integers, $1 + 4n = m^2$ for some positive integer m. Since $1 + 4n$ is odd, m must be odd, i.e., $m = 2k + 1$ for some integer k. Every odd square is of the form $1 + 4n$ since

$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1.$$

Since $1 \leq n \leq 100$, the answer is the number of odd squares between $5 = 1+4\cdot1$ and $401 = 1+4\cdot100$. There are nine odd squares in that range, $3^2, 5^2, 7^2, 9^2, 11^2, 13^2, 15^2, 17^2, 19^2$.

17. On $\triangle ABC$, point D lies on the segment AB, point E lies on BC, and point F lies on CA. If $\frac{AD}{DB} = \frac{BE}{EC} = \frac{CF}{FA} = \frac{2}{3}$, and the area of $\triangle ABC$ equals 1, then what is the area of $\triangle DEF$? Note that the figure below is not drawn to the scale.



(a) $\frac{5}{9}$ **(b)** $\frac{4}{9}$ **(c)** $\frac{3}{25}$ **(d)** $\frac{7}{25}$ **(e)** $\frac{6}{25}$

The answer is: **(d)**

Solution: We compare first the area $\mathcal{A}$ of $\triangle ADF$ and the area $\mathcal{A}'$ of $\triangle ABC$, and we use that $\mathcal{A}' = 1$. Let h denote the length of the altitude FF' drawn from F in the first triangle and h' the length of the altitude CC' drawn from C in the second triangle. Then $\triangle FAF'$ is similar to $\triangle CAC'$. Since $CF/FA = 2/3$, we see that $FA = (3/5)CA$. Hence, $h = (3/5)h'$. Also, $AD/DB = 2/3$ implies $AD = (2/5)AB$. It follows that $\mathcal{A} = (1/2)h(AD) = (6/25)(1/2)h'(AB) = (6/25)\mathcal{A}' = 6/25$. Similarly, we deduce that each of $\triangle EDB$ and $\triangle CFE$ has area $6/25$. It follows that $\triangle DEF$ has area $1 - 3(6/25) = 7/25$.

18. When Alice entered the Forest of Forgetfulness, she forgot the day of the week. She met the Lion and the Unicorn resting under a tree. The Lion lies on Mondays, Tuesdays and Wednesdays and tells the truth on the other days of the week. The

Unicorn, on the other hand, lies on Thursdays, Fridays, and Saturdays, but tells the truth on the other days of the week. They made the following statements:

Lion: "Yesterday was one of my lying days."

Unicorn: "Yesterday was one of my lying days."

From these two statements, Alice was able to deduce the day of the week. What day was it?

(a) Monday **(b)** Wednesday **(c)** Thursday **(d)** Friday
(e) Sunday

The answer is: **(c)**

Solution: If the Lion is telling the truth, the day of the week must be Thursday. If he is lying, then the day of the week must be Monday. So the day of the week must be either Thursday or Monday. If the Unicorn is telling the truth, the day of the week must be Sunday. If he is lying, then the day of the week must be Thursday. The day of the week cannot be Sunday (since we have already said that it must be Thursday or Monday). Therefore, it must be Thursday.

19. Suppose that $f(x) = ax^2 - \sqrt{2}$ for some positive real number a.
If $f(f(\sqrt{2})) = -\sqrt{2}$, then what is a equal to?

(a) $\frac{\sqrt{2}}{2}$ **(b)** $\frac{2+\sqrt{2}}{2}$ **(c)** $\frac{2-\sqrt{2}}{2}$ **(d)** $2 - \sqrt{2}$ **(e)** none of the above

The answer is: **(a)**

Solution: $f(f(\sqrt{2})) = a(2a - \sqrt{2})^2 - \sqrt{2}$ which we set equal to $-\sqrt{2}$. Thus $a(2a - \sqrt{2})^2 = 0$. Since $a > 0$, we have $2a - \sqrt{2} = 0$ and hence $a = \sqrt{2}/2$.

20. From a class of 12 students, 3 are chosen to form a math contest team. The team is required to include at least one boy and at least one girl. If exactly 160 different teams can be formed from the 12 students, then which of the following can be the difference between the number of boys and the number of girls in the class?

(a) 0 **(b)** 2 **(c)** 4 **(d)** 6 **(e)** 8

The answer is: **(c)**

Solution: If g is the number of girls and b is the number of boys, then $g + b = 12$. A team can be chosen by selecting one of the g girls, one of the b boys, and one of the remaining $g + b - 2 = 10$ students. If we consider every such formulation of a team, we will count each team twice (for example, if the first person selected is Bing and the second person selected is Curt and the third person selected is Dave, this will result in the same team as selecting first Bing, then Dave, and then Curt).

Hence, the total number of teams possible is $5gb$. Since this total is 160, we obtain that $gb = 32$. Given $g + b = 12$ and $gb = 32$, we get that g and b are 8 and 4 in some order. The difference is then the answer indicated.

21. An integer valued point in the xy plane is a point (a, b) where both a and b are integers. How many integer valued points are on or inside a circle of radius 4 centered at the origin?

(a) 48 **(b)** 29 **(c)** 60 **(d)** 31 **(e)** none of the above

The answer is: **(e)**

Solution: We are counting the points (x, y) where x and y are integers satisfying the

$$\sqrt{x^2 + y^2} \leq 4.$$

One can run through the various possibilities:
- if $y = 0$, then x can be any number from -4 to 4 so this is 9 points.
- if $y = \pm 1$, then there are 7 possibilities for x ranging from -3 to 3.
- if $y = \pm 2$, then there are still 7 possibilities for x ranging from -3 to 3.
- if $y = \pm 3$, then there are 5 possibilities for x ranging from -2 to 2.
- and finally when $y = \pm 4$ the only possible value for x is 0.

Adding all the possibilities up gives

$$9 + 7 + 7 + 7 + 7 + 5 + 5 + 1 + 1 = 49.$$

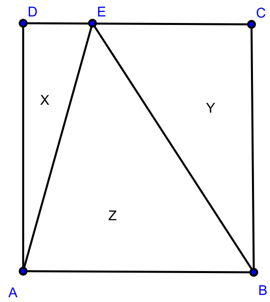
22. Consider the points $A(-5, -1), B(-1, 0), C(1, 2)$, and $D(1, 3)$. Let P be a point and let $d = PA^2 + PB^2 + PC^2 + PD^2$ so that d is the sum of the squares of the distances from P to each of A, B, C, and D. What is the least possible value for d?

(a) 30 **(b)** 42 **(c)** 36 **(d)** 38 **(e)** 34

The answer is: **(e)**

Solution: Let $P = (x, y)$. Then
$d = (x+5)^2 + (y+1)^2 + (x+1)^2 + y^2 + (x-1)^2 + (y-2)^2 + (x-1)^2 + (y-3)^2$, i.e.,
$d = 4x^2 + 8x + 4y^2 - 8y + 42 = 4(x+1)^2 + 4(y-1)^2 + 34$.
Since $(x+1)^2 \geq 0$ for all x with equality precisely when $x = -1$ and since $(y-1)^2 \geq 0$ for all y with equality precisely when $y = 1$, the minimum value of d is 34 (obtained when $x = -1$ and $y = 1$).

23. Suppose that $ABCD$ is a rectangle, and that E is a point on CD. Let X be the area of $\triangle AED$, Y be the area of $\triangle BCE$, and Z be the area of $\triangle ABE$, and suppose that $Y^2 = XZ$. What is the value of $\frac{DE}{EC}$?



(a) $\frac{\sqrt{5}}{3}$ **(b)** $\frac{-1+\sqrt{5}}{2}$ **(c)** $\frac{\sqrt{3}}{2}$ **(d)** $\frac{3}{5}$ **(e)** none of the above

The answer is: **(b)**

Solution: We use that $(2Y)^2 = (2X)(2Z)$ and $AD = BC$. Since $2Y$ is twice the area for $\triangle BCE$, we deduce $2Y = EC \times BC$. Similarly, $2X = DE \times AD = DE \times BC$ and $2Z = (DE + EC) \times BC$. Hence, $EC^2BC^2 = DE \times BC \times (DE + EC) \times BC$ so that $EC^2 = DE^2 + DE \times EC$. Dividing by EC^2, we see that DE/EC is a solution of $1 = x^2 + x$. By the Quadratic Formula, $x^2 + x - 1$ has $(-1 + \sqrt{5})/2$ as its only positive root. Hence, $DE/EC = (-1 + \sqrt{5})/2$.

24. The area of a square with diagonal $\sqrt{8}$ is

(a) 8 **(b)** $2\sqrt{2}$ **(c)** 4 **(d)** $\sqrt{2}$ **(e)** none of the above

The answer is: **(c)**

Solution: If the square has side c, then by pythagorean theorem, $2c^2 = 8$ (as the diagonal is the hypothenuse of the isosceles right triangle form by two consecutive sides of the triangle). Hence $c^2 = 4$ which is the area of the square.

25. For how many integers m, with $10 \le m \le 100$, is $m^2 + m - 90$ divisible by 17 ?

(a) 7 **(b)** 8 **(c)** 9 **(d)** 10 **(e)** none of the above

The answer is: **(d)**

Solution: Since $m^2+m-90 = (m-9)(m+10)$, we have that 17 divides m^2+m-90 if and only if 17 divides $m-9$ or $m+10$. The m with $10 \le m \le 100$ such that $m-9$ is divisible by 17 are 26, 43, 60, 77, and 94. The m with $10 \le m \le 100$ such that $m+10$ is divisible by 17 are 24, 41, 58, 75, and 92. Hence, the answer is 10.

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Dear Teachers/Students:

If you do have any suggestions about the competition, or if you have different solutions to any of this year's problems, please send them by mail or e-mail to

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Remember to visit us for information about past competitions at
http://www.svsu.edu/matholympics/

The SVSU Math Olympic Committee would like to express his gratitude to all participants.

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