SAGINAW VALLEY STATE UNIVERSITY
2012 MATH OLYMPICS LEVEL II

1. If $T_n = 1 + 2 + 3 + \ldots + n$ and

$$P_n = \frac{T_2}{T_2 - 1} \cdot \frac{T_3}{T_3 - 1} \cdot \frac{T_4}{T_4 - 1} \cdot \quad \cdots \quad \cdot \frac{T_n}{T_n - 1}.$$

for $n = 2, 3, 4, \ldots$, then P_{2012} is the closest to which of the following numbers?

(a) 2.9 **(b)** 2.3 **(c)** 3.1 **(d)** 2.6 **(e)** 3.5

2. Find the value of the sum $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)}$.

(a) $\frac{99}{100}$ **(b)** $\frac{1}{10100}$ **(c)** $\frac{100}{101}$ **(d)** $\frac{194}{100}$ **(e)** none of the above

3. What is the last digit of 3^{2012}?

(a) 1 **(b)** 3 **(c)** 7 **(d)** 9 **(e)** none of the above

4. Which of the following is equivalent to $\tan(\frac{1}{2}\cos^{-1} x)$, for $-1 < x \leq 1$?

(a) $\frac{\sqrt{2}}{\sqrt{x^2+4}}$ **(b)** $\frac{\sqrt{1-x^2}}{1+x}$ **(c)** $\frac{\sqrt{4-x^2}}{x}$ **(d)** $\pm\sqrt{\frac{1+x^2+\sqrt{1-x^2}}{2(1+x^2)}}$

(e) none of the above

5. Suppose a group of people have a code between themselves on how they can send messages to others in the group.

• For each subgroup of distinct people A, B and C in the group the following holds: if A can send a message to B and B can send a message to C, then C can send a message to A.

• For each pair of distinct people A and B in the group, either A can send a message to B or B can send a message to A but not both.

What is the largest number of people in the group?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** none of the above

6. A problem to remember the year 2011: how many positive integers less than or equal to 2011 are multiples of both 3 and 5 but not multiples of 8?

(a) 110 **(b)** 118 **(c)** none **(d)** 30 **(e)** none of the above

7. At an artisan bakery, French tortes are 52 dollars each, almond tarts are 12 dollars each and cookies are one dollar each. If Alex has 400 dollars to purchase exactly 100 of these items for a party and he buys at least one of each item, how many French Tortes does he purchase? Only whole pieces of the bakery items can be purchased.

(a) 4 **(b)** 5 **(c)** 3 **(d)** 1 **(e)** 2

8. We are given 6 points on a circle equally spaced (think of a clock with just the even hours). How many triangles can be constructed so that their vertices are three of the given points on the circle? How many among those triangles are isosceles triangles (isosceles means the triangle has at least two equal sides)?

(a) 20 triangles, 6 isosceles **(b)** 18 triangles, 8 isosceles **(c)** 20 triangles, 8 isosceles
(d) 18 triangles, 6 isosceles **(e)** none of the above

9. $\cos 3\theta = ?$

(a) $4\cos^3\theta - 3\cos\theta$ **(b)** $2(\cos^2\theta - \sin^2\theta)$ **(c)** $\cos\theta(1 + 2\sin^2\theta)$
(d) $4\cos^3\theta - \cos\theta$ **(e)** none of the above

10. What is the value of the following product $\sin\frac{\pi}{32}\cos\frac{\pi}{32}\cos\frac{\pi}{16}\cos\frac{\pi}{8}\cos\frac{\pi}{4}$?

(a) $\frac{1}{2}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{8}$ **(d)** $\frac{1}{16}$ **(e)** none of the above

11. What is the remainder when $P(x) = 1 - x + 2x^4 - 3x^9 + 4x^{16} - 5x^{25} + x^{2011} + 6x^{2012}$ is divided by $D(x) = x^2 - 1$?

(a) 15 **(b)** $-2x + 3$ **(c)** $-8x + 13$ **(d)** $x^2 - x + 1$
(e) none of the above

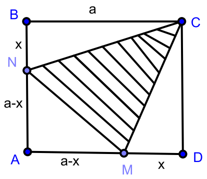
12. Let $\triangle ABC$ be a right triangle ($\angle C = 90°$) such that $AB = 25$ cm, $BC > AC$ and the radius r of the inscribed circle in $\triangle ABC$ is 5 cm. Let also CD be the height of $\triangle ABC$ from the vertex C to the side AB. Find the radius of the circle inscribed in $\triangle CDM$ where M is the midpoint of the side BC.

(a) 3 cm **(b)** 2 cm **(c)** 4 cm **(d)** 5 cm **(e)** none of the above

13. The triangle $\triangle ABC$ has $AB = 7$ and the lengths of the other two sides have given ratio $BC/CA = 24/25$. What is the largest possible area for the $\triangle ABC$?

(a) $25\sqrt{2}$ **(b)** 300 **(c)** 12 **(d)** $5\sqrt{2}$ **(e)** none of the above

14. An equilateral triangle is inscribed in a square with side a so that one of the vertices of the triangle coincides with one of the vertices of the square. Find the area of the triangle in terms of a.



(a) $(2\sqrt{2}-3)a^2$ **(b)** $(2\sqrt{3}-3)a^2$ **(c)** $(2\sqrt{5}-3)a^2$ **(d)** $(2\sqrt{5}-4)a^2$
(e) none of the above

15. Horses A, B and C are entered in a three-horse race in which ties are not possible. If the odds against A winning are 3-to-1 and the odds against B winning are 2-to-3, what are the odds against C winning? (By "odds against X winning are p-to-q" we mean that the probability of X winning the race is $\frac{q}{p+q}$.)

(a) 3-to-20 **(b)** 5-to-6 **(c)** 8-to-5 **(d)** 17-to-3 **(e)** 20-to-3

16. A frog makes 2 jumps, each 1 meter in length. The direction of the jumps are chosen independently and at random in any direction in a plane. What is the probability that the frogs final position is at most 1 meter from its starting position?

(a) $\frac{1}{2}$ **(b)** $\frac{2}{3}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{4}$ **(e)** none of the above

17. Let z_1, z_2 and z_3 be the roots of the polynomial $Q(x) = x^3 - 9x^2 + 1$. In other words, $Q(z_1) = Q(z_2) = Q(z_3) = 0$. If $P(x) = x^5 - x^2 - x$, what is the value of $P(z_1) + P(z_2) + P(z_3)$?

(a) 0 **(b)** $153,300$ **(c)** $13,320$ **(d)** $58,554$ **(e)** none of the above

18. $\frac{\frac{1}{5}+1}{\frac{1+\sqrt[3]{5}}{5}}$ is equal to which of the following numbers?

(a) $\sqrt[3]{5}-1$ **(b)** $6(1+\sqrt[3]{5})$ **(c)** $\frac{-3(1+\sqrt[3]{5})}{2}$ **(d)** $1-\sqrt[3]{5}+\sqrt[3]{25}$
(e) none of the above

19. For each real number x , let $g(x)$ be the minimum value of the numbers $6x+3, 2x+7, 15-x$. (For example if $x = 2$, then the three numbers are 15,11,13, so $g(2) = 11$.) What is the maximum value of $g(x)$?

(a) $\frac{125}{9}$ **(b)** $\frac{40}{3}$ **(c)** $\frac{109}{9}$ **(d)** $\frac{37}{3}$ **(e)** none of the above

20. Let f be defined by $f(0) = 1$ and $f(x + \frac{1}{2}) = 2f(x) - \frac{1}{2}$. What is $f(-1)$?

(a) $\frac{3}{4}$ **(b)** $\frac{5}{8}$ **(c)** 2 **(d)** Cannot be determined from the given information
(e) none of the above

21. If z is a complex number satisfying $z^5 = 1$, but $z^4 \neq 1$, what is $z^4 + z^3 + z^2 + z$?

(a) -1 **(b)** 2 **(c)** 0 **(d)** i **(e)** none of the above

22. An equilateral triangle $\triangle ABC$ with a side a is revolved about a straight line l through B that doesn't intersect the triangle anywhere else. The angle α betweenAB and l is between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Find the surface area of the solid obtained by rotating $\triangle ABC$ about l.

(a) $\sqrt{3}\pi a^2 \sin\alpha$ **(b)** $2\sqrt{3}\pi a^2 \sin(30° - \alpha)$ **(c)** $2\sqrt{3}\pi a^2 \cos(60° - \alpha)$
(d) $\sqrt{3}\pi a^2 \cos\alpha$ **(e)** none of the above

23. Find the smallest positive integer n such that every digit of $45n$ is 0 or 4.

(a) $n = 23456789$ **(b)** $n = 123456789$ **(c)** $n = 987654321$ **(d)** $n = 98765432$
(e) none of the above

24. Thirty bored students take turns walking down a hall that contains a row of closed lockers, numbered 1 to 30. The first student opens all the lockers; the second student closes all the lockers numbered 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30; the third student operates on the lockers numbered 3, 6, 9, 12, 15, 18, 21, 24, 27, 30: if a locker was closed, he opens it, and if a locker was open, he closes it; and so on. For the i^{th} student, he works on the lockers numbered by multiples of i: if a locker was closed, he opens it, and if a locker was open, he closes it. What is the number of lockers that remain open after all the students finish their walks?

(a) 2 **(b)** 4 **(c)** 5 **(d)** 12 **(e)** none of the above

25. Let n and d be fixed integers. How many integers m are such that $n^2 + md$ is an exact square?

(a) Infinitely many m **(b)** 255 **(c)** 625 **(d)** no such m exists
(e) none of the above