Saginaw Valley State University

2009 Math Olympics – Level II Solutions

1. $f(x)$ is a degree three monic polynomial (leading coefficient is 1) such that $f(0) = 3$, $f(1) = 5$ and $f(2) = 11$. What is $f(5)$?

(a) 27 **(b)** 113 **(c)** 126 **(d)** 173 **(e)** None of the above

SOLUTION **(b)**: $f(x) = x^3 + ax^2 + bx + c$. We know $c = 3$ from $f(0) = 3$. From $f(1) = 5$ and $f(2) = 11$ we have $1 + a + b + 3 = 5$ and $8 + a4 + b2 + 3 = 11$, giving the system of equations: $a + b = 1$ and $4a + 2b = 0$. Solving this system gives $a = -1$, $b = 2$. So $f(x) = x^3 - x^2 + 2x + 3$. So $f(5) = 125 - 25 + 10 + 3 = 113$.

2. Which of the following are equal to $\ln \frac{\sqrt{2}+1}{\sqrt{2}-1}$?

(a) $2\ln\left(1+\sqrt{2}\right)$ **(b)** $-\ln\left(1-\sqrt{2}\right)^2$ **(c)** $\ln\left(2\sqrt{2}+3\right)$ **(d)** All of the above

(e) None of the above

SOLUTION **(d)**: Since $\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{(\sqrt{2}+1)^2}{2-1} = \left(\sqrt{2}+1\right)^2$, using properties of logarithms, $\ln \frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln\left(\sqrt{2}+1\right)^2 = 2\ln\left(1+\sqrt{2}\right)$. Also, $\left(\sqrt{2}+1\right)^2 = 2 + 2\sqrt{2} + 1 = \sqrt{2} + 3$. Finally, $\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{(\sqrt{2}+1)(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}-1)} = \frac{2-1}{(\sqrt{2}-1)^2} = \frac{1}{(1-\sqrt{2})^2}$.
Using properties of logarithms again, $\ln \frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln \frac{1}{(1-\sqrt{2})^2} = -\ln\left(1-\sqrt{2}\right)^2$.

3. The symbol R_k stands for a positive integer whose base-ten representation is a sequence of k ones, that is $R_1 = 1$, $R_2 = 11$, $R_3 = 111$, etc. The quotient $\frac{R_{24}}{R_4}$ is an integer whose base-ten representation contains only digits 0 and 1. The number of digits 0 in this representation is:

(a) 6 **(b)** 11 **(c)** 15 **(d)** 16 **(e)** 20

SOLUTION **(c)**: Using a decimal expansion, $R_k = 1 + 10 + 10^2 + 10^3 + \ldots + 10^{k-1}$. This is a finite geometric series with common ratio 10 and first term 1. Therefore

$$R_k = \frac{1-10^k}{1-10} = \frac{10^k-1}{9}$$

and

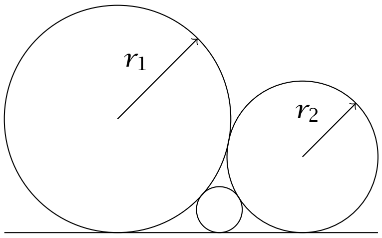
$$\frac{R_{24}}{R_4} = \frac{\frac{10^{24}-1}{9}}{\frac{10^4-1}{9}} = \frac{10^{24}-1}{10^4-1} = \frac{\left(10^4\right)^6-1}{10^4-1}$$

The last is the sum of the first six terms of the geometric series with first term 1 and common ratio 10^4, which means

$$\frac{R_{24}}{R_4} = 1 + 10^4 + 10^8 + 10^{12} + 10^{16} + 10^{20} = 100,010,001,000,100,010,001$$

which has 15 zeros.

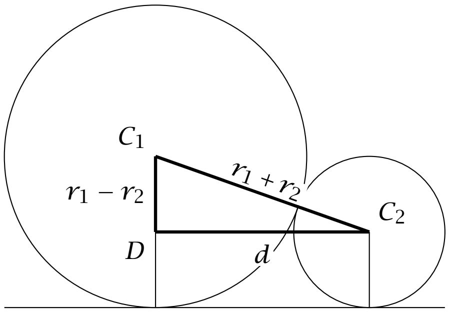
4. Three circles are tangent to each other and to a straight line, as shown in the picture. Express the radius r of the smallest circle in terms of radii r_1 and r_2 of the other two circles.



(a) $r = \frac{r_1 r_2}{r_1+2\sqrt{r_1 r_2}+r_2}$ **(b)** $r = \frac{r_1 r_2}{r_1-2\sqrt{r_1 r_2}+r_2}$

(c) $r = \frac{r_1 r_2}{r_1+r_2}$ **(d)** $r = r_1 - r_2$

(e) None of the above



Solution **(a)**: Start with two circles with a common tangent, as in the picture on the left. The right triangle $\triangle C_1C_2D$ has hypotenuse $r_1 + r_2$ and legs $r_1 - r_2$ and d, where d is also the distance between the points where the tangent line touches the circles.

From the Pythagorean Theorem, $d = \sqrt{(r_1+r_2)^2 - (r_1-r_2)^2} = \sqrt{4r_1r_2} = 2\sqrt{r_1r_2}$.

Applying this result to the three pairs of given circles, we get

$$2\sqrt{r_1r_2} = 2\sqrt{r_1r} + 2\sqrt{rr_2}$$

Solving this equation for r we first obtain

$$\sqrt{r} = \frac{\sqrt{r_1r_2}}{\sqrt{r_1}+\sqrt{r_2}} \quad \text{and} \quad r = \frac{r_1r_2}{(\sqrt{r_1}+\sqrt{r_2})^2} = \frac{r_1r_2}{r_1+2\sqrt{r_1r_2}+r_2}.$$

5. A box contains 4 red balls and 6 white balls. A sample of size 3 is drawn without replacement from the box. What is the probability of obtaining 1 red ball and 2 white balls, if you know that at least 2 of the balls in the sample are white?

(a) 1/2 **(b)** 2/3 **(c)** 3/4 **(d)** 9/11 **(e)** 54/55

Solution **(c)**: This is a conditional probability:

$$P(\text{one red and two white}|\text{at least two white}) = \frac{P(\text{one red and two white})}{P(\text{at least two white})}$$

Then

$$\begin{aligned} P(\text{one red and two white}) &= P(\text{red, white, white}) + P(\text{white, red, white}) + P(\text{white, white, red}) \\ &= \frac{4}{10}\cdot\frac{6}{9}\cdot\frac{5}{8} + \frac{6}{10}\cdot\frac{4}{9}\cdot\frac{5}{8} + \frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} = 3\frac{6\cdot5\cdot4}{10\cdot9\cdot8} = \frac{1}{2} \end{aligned}$$

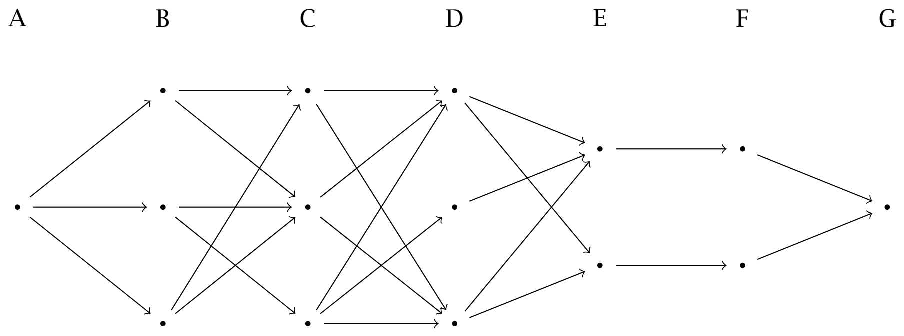
The probability of at least two white is

$$P(\text{one red and two white}) + P(\text{three whites}) = \frac{1}{2} + \frac{6}{10}\frac{5}{9}\frac{4}{8} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

Finally

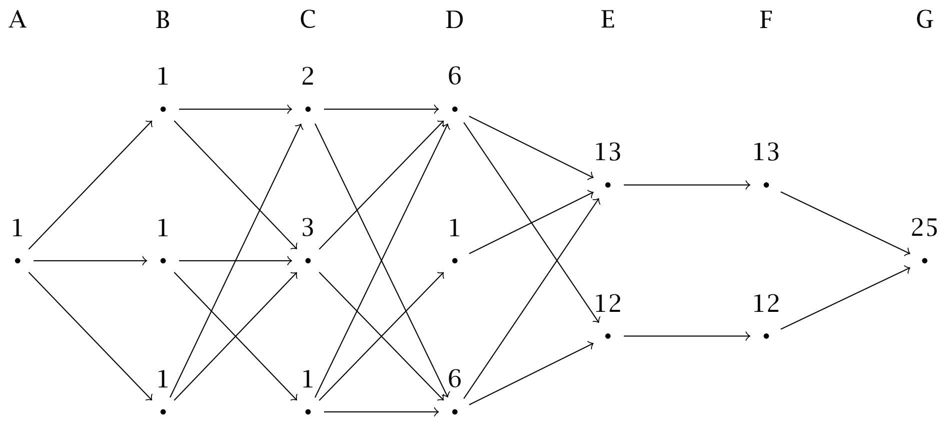
$$P(\text{one red and two white}|\text{at least two white}) = \frac{1/2}{2/3} = \frac{3}{4}.$$

6. In the diagram, A, B, ..., G refer to successive states through which a traveler must pass in order to get from A to G, moving from left to right. A path consists of a sequence of line segments leading from one state to the next. A path must always move to the next state until reaching state G. Determine the number of possible paths from A to G.



(a) 20 **(b)** 23 **(c)** 24 **(d)** 25 **(e)** 30

Solution **(d)**: We will start at the state A, and going from left to right, for each node in the diagram we will mark in how many different ways can one get to that node from A. For each of the nodes in state B, there is exactly one way to get there. There are two arrows leading to the topmost node in the state C, each coming from a node that has exactly one way of getting to it. So there are $1 + 1 = 2$ different ways of getting to this node. There are $1 + 1 + 1 = 3$ ways to get to the middle node in state C, and only 1 way to get to the lowest node in the state C. There are 3 arrows ending at the topmost node in the state D. The first one arriving from a node which has 2 ways of getting to it, the second from a node that has 3 ways of getting to it, and the last from a node that has only 1 way to get to it. So altogether there are $2+3+1 = 6$ ways of getting to the topmost node in the state D. Similarly, there is only 1 way to get to the middle node in the state D, and 6 ways to get to the lowest node of the state D. The emerging pattern is: the label for each node will be the sum of the labels at the beginning of all the arrows that end in the given node. So the two nodes in state E will be labeled $6 + 1 + 6 = 13$ and $6 + 6 = 12$. Each of the nodes in state F have only one arrow ending in them, so the labels will also be 13 and 12, respectively. Finally, the last node will have $13 + 12 = 25$ different ways to get to it from the initial node in the state A.



7. In the diagram, the number in the first box is 4 and the number in the second box is 7. Beginning with the second box, the number in each subsequent box is the sum of the numbers in the two boxes immediately adjacent to its box on each side. What is the number in the 2009th box?

4	7		$\cdots$

(a) -3 **(b)** 4 **(c)** -7 **(d)** 7 **(e)** Not enough information given

Solution **(c)**: In order to have 7 in the second box, the third box must have $7-4=3$. The fourth box will have $3-7=-4$, the fifth box $-4-3=-7$, the sixth box $-7-(-4)=-3$, the seventh box $-3-(-7)=4$, the eight box $4-(-3)=7$. Since the value of each box is determined by the values in the previous two boxes, we see that the sequence repeats. So the sequence will look like

$$4, 7, 3, -4, -7, -3, 4, 7, 3, -4, -7, -3, 4, 7, 3, \ldots$$

There are 6 repeating numbers. Since $2009 \bmod 6 = 5$, the number in the 2009th box will be -7.

8. The Awesome Mathematical Machine is capable of two operations. It can increase a number by one, or multiply it by two. The number 0 was entered into the machine. After a while, the machine came up with the result of 2009. What is the smallest number of steps the machine had to perform in order to obtain this result?

(a) 13 **(b)** 18 **(c)** 21 **(d)** 103 **(e)** None of the above

Solution **(b)**: With the exception of 0 and 1, multiplying by 2 leads to a bigger gain than adding 1. So we want to do as many multiplications by 2 as possible, and only add 1 if we have to.
We will start with the resulting number and figure out which operations are needed to obtain it.
Since 2009 is odd, it cannot be a result of multiplication by 2, so it must have been obtained from 2008 by adding 1. The number 2008 is even, so could be obtained from 1004 by multiplying by 2. The number 1004 could be obtained from 502 by multiplying by 2, and 502 could be obtained from 251 by multiplying by 2. The number 251 is odd so it must have been obtained by adding 1 to 250 etc.
The sequence of operations when written backwards from last to first is then

$$2009 \xleftarrow{+1} 2008 \xleftarrow{\times 2} 1004 \xleftarrow{\times 2} 502 \xleftarrow{\times 2} 251 \xleftarrow{+1} 250 \xleftarrow{\times 2} 125 \xleftarrow{+1} 124 \xleftarrow{\times 2} 62 \xleftarrow{\times 2} 31 \xleftarrow{+1} 30$$
$$\xleftarrow{\times 2} 15 \xleftarrow{+1} 14 \xleftarrow{\times 2} 7 \xleftarrow{+1} 6 \xleftarrow{\times 2} 3 \xleftarrow{+1} 2 \xleftarrow{\times 2} 1 \xleftarrow{+1} 0$$

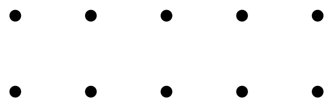
There were 18 operations involved.

9. January 1980 had exactly 4 Mondays and 4 Fridays. Which day of the week was January 1st?

(a) Monday **(b)** Tuesday **(c)** Thursday **(d)** Saturday **(e)** Impossible to determine

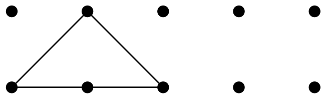
Solution **(b)**: January has 31 days. There are 4 complete weeks, and 3 additional days. If January starts on Saturday, Sunday or Monday, there is one more Monday at the end of the month. If January starts on Wednesday, Thursday of Friday, there is one additional Friday at the end of the month. The only day of the week which will result in 4 Mondays and 4 Fridays is Tuesday.

10. The points on the right form a square grid. How many isosceles triangles can be drawn with vertices at the points?

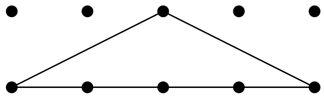


(a) 8 **(b)** 16 **(c)** 24 **(d)** 32 **(e)** 40

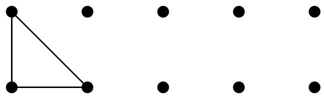
SOLUTION **(c)**: There will be 3 triangles of this type:



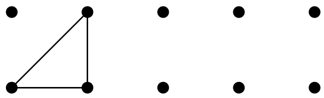
and one triangle like this:



Another 4 triangles can be obtained from the above by turning them upside down. There will be 4 triangles like this one:

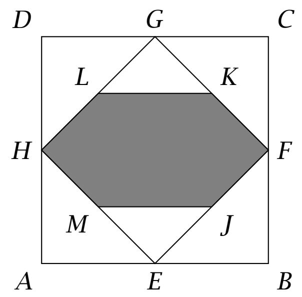


and 4 like this:

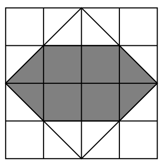


and 8 more like the 8 previous ones turned upside down.

11. The area of the square $ABCD$ is 64. The midpoints of its sides are joined to form the square $EFGH$. The midpoints of its sides are J, K, L and M. The area of the shaded region is



(a) 23 **(b)** 24 **(c)** 20 **(d)** 28 **(e)** 16



SOLUTION **(b)**: The length of each side of square $ABCD$ is 8. We will cut the square into 16 little 2×2 squares, as shown on the left. The shaded region consists of 4 complete squares and 4 half squares, so the shaded area is $4 \cdot 4 + 4 \cdot 2 = 24$.

12. Which of the following describes the locus of all points z in the complex plane such that

$$\left|\frac{z-1}{z}\right| = \sqrt{2}?$$

(a) Vertical line through $x = \sqrt{2} - 1$

(b) Circle with center $(-1, 0)$ and radius $\sqrt{2}$

(c) Circle with center $(-1/2, 0)$ and radius $\frac{\sqrt{2}}{2}$

(d) Hyperbola with foci $(\pm 1, 0)$ and eccentricity $\sqrt{2}$

(e) None of the above

Solution **(b)**: We can rewrite the given equation as

$$|z-1| = \sqrt{2}|z|$$

or, using $z = x + iy$,

$$\sqrt{(x-1)^2 + y^2} = \sqrt{2}\sqrt{x^2 + y^2}.$$

Squaring both sides and simplifying, we get

$$\begin{aligned} x^2 - 2x + 1 + y^2 &= 2x^2 + 2y^2 \\ 1 &= x^2 + 2x + y^2 \\ 2 &= x^2 + 2x + 1 + y^2 \\ 2 &= (x+1)^2 + y^2 \end{aligned}$$

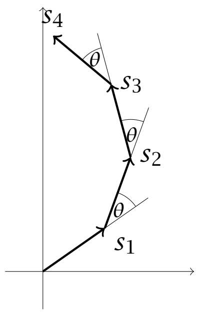
The last equation is an equation of the circle with center at $(-1, 0)$ and radius $\sqrt{2}$.

13. Let $w = 4 + 3i$ and let $z = \frac{w}{|w|}$. For any positive integer n, let

$$s_n = \sum_{k=1}^{n} z^k = z + z^2 + z^3 + \ldots + z^n$$

Then all s_n lie on a circle with radius:

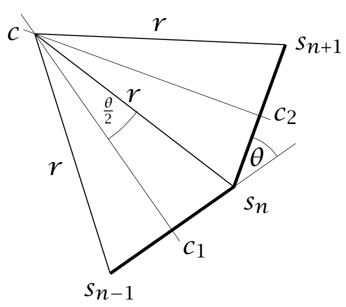
(a) $\frac{\sqrt{5}}{2}$ **(b)** $\frac{3}{2}$ **(c)** $\frac{\sqrt{10}}{2}$ **(d)** $\frac{10}{3\pi}$ **(e)** They do not lie on a circle



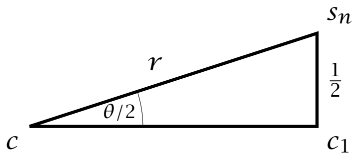
Solution **(c)**: The modulus (absolute value) of w is $\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5$. The modulus of z is 1. We will denote the argument of both z and w by θ. Then $\sin\theta = 3/5$ and $\cos\theta = 4/5$. Then the modulus of z^n is also 1, while the argument of z^n is $n\theta$. In other words, z^{n+1} can be obtained from z^n by rotating it by the angle θ counterclockwise. The picture on the left shows the first few s_n. Each arrow represents z^n for some n.

The perpendicular bisectors of the segments $s_n s_{n+1}$ will all meet at one point[1], which is the center of the circle containing all the s_n. The drawing on the right shows three consecutive s_ns.

[1] Can you explain why?



In the drawing, c is the center of the circle containing the points s_n, the lines cc_1 and cc_2 are the perpendicular bisectors of the segments $s_{n-1}s_n$ and s_ns_{n+1}, respectively. The radii we are trying to find are marked with r. Note that since $|z| = 1$, the distance between s_{n-1} and s_n is 1, and the distance between c_1 and s_n is $1/2$. That will give us the following right triangle:



From here we can easily calculate r:

$$\frac{1/2}{r} = \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-4/5}{2}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}$$

and $r = \sqrt{10}/2$.

14. How many zeros are there at the end of the product of the first 20 primes?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 5 **(e)** None of the above

Solution **(b)**: If there were 2 or more zeros, the product would be divisible by 100. But $100 = 2 \cdot 2 \cdot 5 \cdot 5$, and since in the product of the first 20 primes each prime number appears only once, there has to be less than two zeros. On the other hand, 2 and 5 are both among the first 20 primes, and $2 \cdot 5 = 10$, so the product of the first 20 primes will be divisible by 10 and will have at least 1 zero at the end.

15. How many unique solutions of the equation $\tan 2t - 2\cos t = 0$ are in the interval $[0, 2\pi)$?

(a) none **(b)** 2 **(c)** 4 **(d)** 5

(e) infinitely many

Solution **(c)**: First rewrite the equation:

$$\begin{aligned} \frac{\sin 2t}{\cos 2t} - 2\cos t &= 0 \\ \frac{2\sin t\cos t}{1-2\sin^2 t} - 2\cos t &= 0 \\ 2\cos t\left(\frac{sint}{1-2\sin^2 t} - 1\right) &= 0 \end{aligned}$$

Using the zero product property we get that either $\cos t = 0$ or $\sin t = 1 - 2\sin^2 t$. The first equation has two solutions in $[0, 2\pi)$: $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. The second equation is a quadratic equation in $\sin t$ that can be factored as $(2\sin t - 1)(\sin t + 1) = 0$, which means that $\sin t = 1/2$ or $\sin t = -1$. The only solution of $\sin t = -1$ in the given interval is $\frac{3\pi}{2}$, which is already listed. The equation $\sin t = 1/2$ has two additional solutions, which are not listed yet. So the number of unique solutions is 4.

16. Find the sum of the first 100 common terms in the sequences
17, 21, 25, 29, ...
16, 21, 26, 31, ...

(a) 21500 **(b)** 26350 **(c)** 47850 **(d)** 101100 **(e)** None of the above

SOLUTION **(d)**: The first sequence is $17 + 4k$ for $k = 0, 1, 2, \ldots$.
The second sequence is $16 + 5j$ for $j = 0, 1, 2, \ldots$.
The first common term is 21. The first 100 common terms are

$$21 + 20n, \text{ for } n = 0, 1, 2, \ldots, 99$$

The sum is $100 \cdot 21 + 20(0 + 1 + 2 + \cdots + 99) = 2100 + 20\frac{99\cdot 100}{2} = 2100 + 99000 = 101100$.

17. Simplify

$$\sqrt{13 + 30\sqrt{2 + \sqrt{9 + 4\sqrt{2}}}} - \sqrt{18}.$$

(a) 0 **(b)** 3 **(c)** 5 **(d)** $3\sqrt{2}$ **(e)** None of the above

SOLUTION **(c)**: First, $9 + 4\sqrt{2} = 8 + 4\sqrt{2} + 1 = (2\sqrt{2} + 1)^2$.
Therefore $2 + \sqrt{9 + 4\sqrt{2}} = 2 + (2\sqrt{2} + 1) = 2 + 2\sqrt{2} + 1 = (\sqrt{2} + 1)^2$.
Consequently, $13 + 30\sqrt{2 + \sqrt{9 + 4\sqrt{2}}} = 13 + 30(\sqrt{2} + 1) = 43 + 30\sqrt{2} = 25 + 30\sqrt{2} + 18 = (5 + 3\sqrt{2})^2$.
The original expression will then simplify:

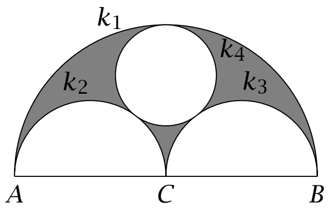
$$\sqrt{13 + 30\sqrt{2 + \sqrt{9 + 4\sqrt{2}}}} - \sqrt{18} = 5 + 3\sqrt{2} - 3\sqrt{2} = 5.$$

18. How many prime numbers less than ten thousand have digits that add up to 2?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

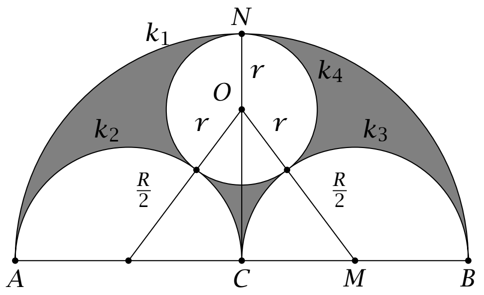
SOLUTION **(c)**: In order for the digits of a number to add up to 2, the digits must be either one 2 and the rest zeros, or two 1s and the rest zeros. Since no prime number can have a 0 in the ones place, and 1 is not considered a prime number, the only candidates are 2, 11, 101 and 1001. Out of those, 2, 11 and 101 are indeed primes, however, $1001 = 910 + 91 = 11 \cdot 91$, so it is not a prime.

19. In the drawing on the right, $AB = 2R$, C is the midpoint of AB, k_1, k_2 and k_3 are semicircles with diameters AB, AC and CB, respectively. The circle k_4 is tangent to k_1, k_2 and k_3. Find the shaded area in terms of R.



(a) $\frac{\pi R^2}{4}$

(b) $\frac{3\pi R^2}{16}$ **(c)** $\frac{5\pi R^2}{36}$ **(d)** $\frac{7\pi R^2}{32}$ **(e)** None of the above



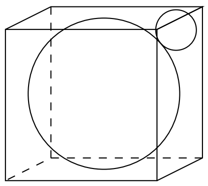
Solution **(c)**: The shaded area is $\frac{1}{2}(\pi R^2) - 2\left(\frac{1}{2}\pi\left(\frac{R}{2}\right)^2\right) - \pi r^2 = \frac{\pi R^2}{4} - \pi r^2$. We need to express r in terms of R. Using the right triangle $\triangle MCO$ and the Pythagorean Theorem, we get

$$\begin{aligned}(R-r)^2 + \left(\frac{R}{2}\right)^2 &= \left(r + \frac{R}{2}\right)^2 \\ R^2 - 2Rr + r^2 + \frac{R^2}{4} &= r^2 + rR + \frac{R^2}{4} \\ R^2 &= 3Rr \\ r &= \frac{R}{3}\end{aligned}$$

Therefore the shaded area is

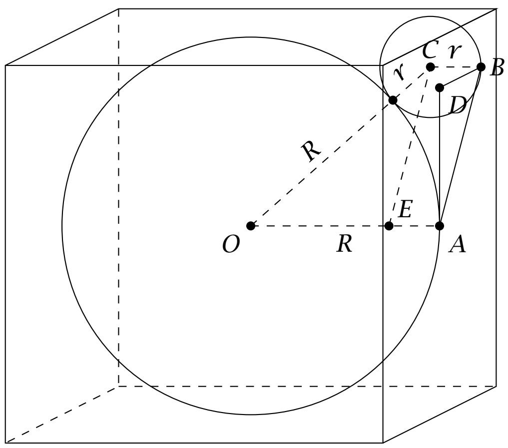
$$\frac{\pi R^2}{4} - \pi\left(\frac{R}{3}\right)^2 = \pi R^2\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5\pi R^2}{36}.$$

20. A sphere with a radius R is inscribed in a cube. Another (smaller) sphere is placed inside a corner of the cube so that it is tangent to three sides of the cube and to the big sphere (see the simplified picture on the right). Find the radius of the small sphere in terms of R.



(a) $r = \frac{(\sqrt{3}-1)R}{2}$ **(b)** $r = (2-\sqrt{3})\,R$ **(c)** $r = \frac{(\sqrt{2}-1)R}{2}$

(d) $r = \left(2-\sqrt{2}\right)R$ **(e)** None of the above



Solution **(b)**: Let's call the radius of the small sphere r. Let A and B be the points of tangency of the larger and smaller spheres, respectively, and the cube. Pick D so that the triangle $\triangle ADB$ is a right isosceles triangle, in which the angle ADB is the right angle, and $|AD| = |DB| = R - r$. Therefore $|AB| = \sqrt{2}(R-r)$. Another right triangle is formed by the points O, E and C, where E is the point on the segment OA with distance r from A, and C is the center of the small sphere. In this right triangle, $|OC| = R + r$, $|OE| = R - r$ and $|CE| = |AB| = \sqrt{2}(R-r)$. According to the Pythagorean Theorem, $|OE|^2 + |EC|^2 = |OC|^2$, or

$$\begin{aligned}(R-r)^2 + \left(\sqrt{2}(R-r)\right)^2 &= (R+r)^2 \\ 3(R-r)^2 &= (R+r)^2 \\ \sqrt{3}(R-r) &= R + r \\ (\sqrt{3}-1)R &= (\sqrt{3}+1)r\end{aligned}$$

which gives us

$$r = \frac{\sqrt{3}-1}{\sqrt{3}+1}R = \frac{(\sqrt{3}-1)^2}{3-1}R = \frac{4-2\sqrt{3}}{2}R = (2-\sqrt{3})R.$$

21. Find $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$.

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** None of the above

SOLUTION **(d)**:

$$\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$$

Rearrange and rewrite

$$\frac{\sin 9^\circ}{\cos 9^\circ} + \frac{\sin 81^\circ}{\cos 81^\circ} - \left(\frac{\sin 27^\circ}{\cos 27^\circ} + \frac{\sin 63^\circ}{\cos 63^\circ}\right)$$

common denominators

$$\frac{\sin 9^\circ \cos 81^\circ + \sin 81^\circ \cos 9^\circ}{\cos 9^\circ \cos 81^\circ} - \frac{\sin 27^\circ \cos 63^\circ + \sin 63^\circ \cos 27^\circ}{\cos 27^\circ \cos 63^\circ}$$

$\sin a \cos b + \cos a \sin b = \sin(a+b)$

$$\frac{\sin 90^\circ}{\cos 9^\circ \cos 81^\circ} - \frac{\sin 90^\circ}{\cos 27^\circ \cos 63^\circ}$$

$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

$$\frac{\sin 90^\circ}{\frac{1}{2}\left[\cos(9^\circ + 81^\circ) + \cos(81^\circ - 9^\circ)\right]} - \frac{\sin 90^\circ}{\frac{1}{2}\left[\cos(27^\circ + 63^\circ) + \cos(63^\circ - 27^\circ)\right]}$$

$\sin 90^\circ = 1$, $\cos 90^\circ = 1$

$$\frac{1}{\frac{1}{2}\cos 72^\circ} - \frac{1}{\frac{1}{2}\cos 36^\circ}$$

common denominator

$$\frac{2\,(\cos 36^\circ - \cos 72^\circ)}{\cos 72^\circ \cos 36^\circ}$$

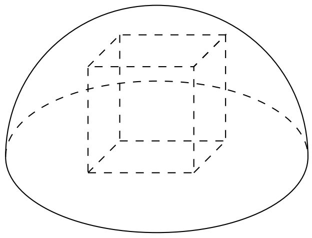
$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$

$$\frac{-4 \sin 54^\circ \sin(-18^\circ)}{\cos 72^\circ \cos 36^\circ}$$

$\sin 54^\circ = \cos 36^\circ$ and $\sin(-18^\circ) = -\cos 72^\circ$

$$4$$

22. A cube is inscribed in a half sphere so that four of its vertices are on the base and the other four are on the sphere. Find the ratio of the volume of the cube to the volume of the half ball.



(a) $\frac{\sqrt{3}}{\pi\sqrt{2}}$ **(b)** $\frac{\sqrt{2}}{\pi\sqrt{3}}$ **(c)** $\frac{\sqrt{3}}{\pi}$ **(d)** $\frac{\sqrt{2}}{\pi}$ **(e)** None of the above

SOLUTION **(b)**: Let r be the radius of the half sphere, and let x be the side of the cube. Then by the Pythagorean Theorem, $x^2 + \left(\frac{\sqrt{2}}{2}x\right)^2 = r^2$, which means that $x = \sqrt{\frac{2}{3}}r$.
The volume of the half sphere is

$$\frac{2\pi r^3}{3}$$

while the volume of the cube is

$$x^3 = \left(\sqrt{\frac{2}{3}}r\right)^3 = \frac{2\sqrt{2}}{3\sqrt{3}}r^3$$

The ratio is then

$$\frac{\frac{2\sqrt{2}}{3\sqrt{3}}r^3}{\frac{2\pi r^3}{3}} = \frac{\sqrt{2}}{\pi\sqrt{3}}.$$

23. If $s = \cos^2\alpha + \cos^2\beta$, find $\cos(\alpha+\beta)\cos(\alpha-\beta)$ in terms of s.

(a) $s-1$ **(b)** $\frac{s-1}{s}$ **(c)** s^2 **(d)** $\sqrt{2}$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta) &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\\ &= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta\\ &= \cos^2\alpha\cos^2\beta - (1-\cos^2\alpha)(1-\cos^2\beta)\\ &= \cos^2\alpha\cos^2\beta - 1 + \cos^2\alpha + \cos^2\beta - \cos^2\alpha\cos^2\beta = -1 + s = s-1.\end{aligned}$$

24. In the expression

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$$

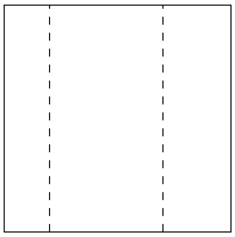
each letter is replaced by one of the digits 1, 2, 3, 4, 5 and 6. What is the largest possible value of an expression that can be obtained in this way.

(a) $8\frac{2}{3}$ **(b)** $9\frac{5}{6}$ **(c)** $9\frac{1}{3}$ **(d)** $9\frac{2}{3}$ **(e)** $10\frac{1}{3}$

SOLUTION **(b)**: To obtain the largest value, we use the larger numbers, 4, 5 and 6, as numerators, and the smaller numbers, 1, 2 and 3 ad denominators. For the first fraction, we can get the largest value if we divide 6 by 1. For the second fraction the largest value will be obtained by dividing 5 by 2, which will then leave us with 4/3. So the largest value is

$$\frac{6}{1} + \frac{5}{2} + \frac{4}{3} = 9\frac{5}{6}.$$

25. A square is cut into three rectangles along two lines parallel to a side, as shown. If the perimeters of of the three rectangles are 22, 26 and 24, respectively, then the area of the original square is



(a) 24 **(b)** 36 **(c)** 64 **(d)** 81 **(e)** 96

SOLUTION **(d)**: Let's call the widths of the three rectangles x, y and z. Then the side of the square is $x+y+z$, and the perimeters of the rectangles are $2x+2(x+y+z)$, $2y+2(x+y+z)$ and $2z+2(x+y+z)$. These have to be equal to 22, 26 and 24. This gives us a system of equations

$$\begin{aligned} 2x + y + z &= 11 \\ x + 2y + z &= 13 \\ x + y + 2z &= 12 \end{aligned}$$

Adding these three equations together gives us $4x+4y+4z = 36$ or $x+y+z = 9$. Therefore the side of the square is 9, and the area is 81.