Saginaw Valley State University
2005 Math Olympics – Level I

1. Which of the following numbers are prime?

   (a) 2783 **(b)** 637 **(c)** 247 **(d)** 359 **(e)** None of the above

   SOLUTION: The first numberm, 2783, is a multiple of 11, 637 is a multiple of 7, and $247 = 19 * 13$. 359 is obviously not divisible by 2, 3 and 5. Since $359 = 357 + 2$, it is not a multiple of 7. Similarly, $359 = 352 + 7$, and as 352 is a multiple of 11 (because $3 + 2 = 5$), 352 cannot be divisible by 11. Also, $359 = 390 - 31$, and 390 is a multiple of 13 while 31 isn't, so 359 is not a multiple of 13. Finally, writing $359 = 340 + 19$ shows that 359 is not divisible by 17. These are all the primes we have to check, since 19^2 is already larger than 359. Therefore 359 is a prime, and the answer is **D**.

2. Simplify the expression $\dfrac{x^2 - y^2}{x^{-1} - y^{-1}}$

   (a) $(x - y)^2(x + y)$ **(b)** $-x^2y - xy^2$ **(c)** 247

   (d) $x + y$ **(e)** None of the above

   SOLUTION: Start by using the difference of squares and the definition of negative exponents to get $\frac{(x-y)(x+y)}{\frac{1}{x}-\frac{1}{y}} = \frac{(x-y)(x+y)}{\frac{y-x}{xy}} = \frac{xy(x-y)(x+y)}{y-x} = xy(x + y)\frac{x-y}{y-x} = xy(x + y)(-1) = -x^2y - xy^2$, so **B** is the correct answer.

3. When going outside on a cold winter day, Jill can choose from three winter coats, five wool scarves, two pairs of boots and four hats. But she will never wear her green scarf unless she wears the blue coat. How many different outfits might her friends see her in?

   (a) 120 **(b)** 104 **(c)** 97 **(d)** 119 **(e)** None of the above

   SOLUTION: With no restrictions on color combinations, there would be $3 \cdot 5 \cdot 2 \cdot 4 = 120$ outfit. However, the outfits with green scarf and any other than blue coat are out of question, so we need to subtract $2 \cdot 1 \cdot 2 \cdot 4 = 16$ outfits. So the answer is **B**, or 104 outfits.

4. Coffee costs about \$6 per pound in the United States. If a Canadian dollar exchanges for 75 U.S. cents, and a kilogram is about 2.2 pounds, what is the equivalent cost in Canadian dollars per kilogram?

(a) \$17.60 **(b)** \$9.90 **(c)** \$3.64 **(d)** \$2.05 **(e)** None of the above

SOLUTION: Let x be the price per pound of coffee in Canadian dollars. Then $x \cdot .75 = 6$, or $x = 8$. So 2.2 pounds will cost $2.2 \cdot 8 = 17.60$ Canadian dollars. **A** is correct.

5. A table in a doll house is a scale model of a full sized table. The surface area of the top of the full sized table is 800 square inches. The surface area of the top of the doll house table is 2 square inches. If the doll house table is 1.5 inches high, how high is the full sized table?

(a) 30 in **(b)** 600 in **(c)** 400 in **(d)** 120 in **(e)** None of the above

SOLUTION: If the ratio of the areas is $800 : 2 = 400 : 1$, the ratio of the lengths will be $\sqrt{400} : \sqrt{1} = 20 : 1$, meaning the full size table is 20 times larger than the doll house table. So the height of the full size table is $20 \cdot 1.5 = 30$ inches. The answer is **A**.

6. Solve the equation. $\left(\frac{1}{3}\right)^{5-x} = 9 \cdot 9^x$

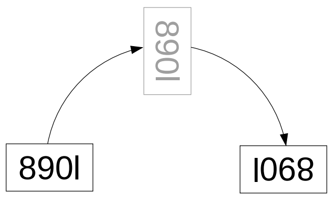
(a) $x = \frac{5}{3}$ **(b)** $x = -7$ **(c)** no solution **(d)** $x = -5$

(e) None of the above

SOLUTION: Using properties of exponents, the left side can be written as $3^{-(5-x)}$, while the right side as 3^{2+2x}. Since $y = 3^x$ is a one-to-one function, this means that $-(5-x) = 2 + 2x$, or $x = -7$. **B** is correct.

7. Assume that the digits 1, 0, and 8 are written so that they look the same when they are upside down as they do when they're right-side up. Assume that the digits 6 and 9 are written so that an upside down 6 is identical to a 9. How many 5 digit numbers look the same upside down as they do right-side up.

A 4-digit example:



(a) 75 **(b)** 868 **(c)** 152 **(d)** 48

(e) None of the above

SOLUTION: The digit in the middle must be one of 0, 1, and 8. So we have three possibilities for the middle digit. When you turn a number upside down, the first digit will become the last and vice versa, so if the first digit is 1, the last will also have to be one, if the first is 8, the last also has to be 8, if the first is 9, the last has to be 6, and if the first is 6, the last has to be 9. The first digit cannot be 0, so that will give us 4 options for the combination of first and last digit. The same is true about second digit and fourth digit, except that we can also use 0 here, so we have 5 options for the combination of these two digits. So all together we have $3 \cdot 4 \cdot 5 = 60$ numbers. So the answer is **E**, none of the above.

8. Suppose you are in a class of 20 students. If every student shakes hands with every other student, how many handshakes will there be?

   (a) 190 **(b)** 400 **(c)** 380 **(d)** 2019 **(e)** None of the above

   SOLUTION: Each of the 20 students will shake hands with 19 people. But that way, each handshake will be counted twice, so the number of handshakes is $\frac{20 \cdot 19}{2} = 190$.

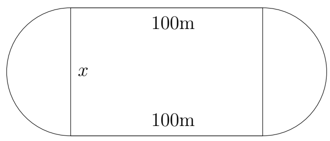
   Another way: The first student shakes hands with 19 people. The second will shake hands with 18 people (not counting the first one again), the third with 17 people etc. So there will be
   $$19 + 18 + 17 + \cdots + 2 + 1 + 0 = \frac{20 \cdot 19}{2}$$
   handshakes. Therefore **A**.

9. A school has three clubs: cheerleaders, pep club and student council. Half of the student council members and all of the cheerleaders belong to the pep club. Two students are members of all three clubs. If there are 24 student council members, 6 cheerleaders, and 40 pep club members, how many students belong only to the pep club?

   (a) 10 **(b)** 24 **(c)** 20 **(d)** 22 **(e)** None of the above

   SOLUTION: There is 12 students who are both on the council and in the pep club, and 6 students who are both cheerleaders and in the pep club. out of those, 2 students are both in the former 12 and the later 6, so there is actually $12 + 6 - 2 = 16$ students who are both in the pep club and at least one other club. So there is $40 - 16 = 24$ students only in the pep club. The answer is **B**.

10. A 400 meter track is to built with 100-meter straight-aways on both sides and semicircular ends. (See the diagram.) How many meters wide should the track be between the straight-aways so that the total perimeter is 400 meters?

    (a) $100/\pi$ **(b)** 100 **(c)** $200/\pi$ **(d)** $50/\pi$ **(e)** None of the above

    SOLUTION: This means that the track consists of a complete circle with diameter x and two straight paths, each of length 100m. So the length is $\pi \cdot x + 200 = 400$, and so $x = 200/\pi$ meters. **C** is correct.

11. Find the perimeter of a rectangle whose area is 22 square meters and whose diagonal is 10 meters

    (a) 24 **(b)** 12 **(c)** 44 **(d)** 33 **(e)** None of the above

    SOLUTION: If the dimensions of the rectangle are xm and ym, the perimeter will be $2(x+y)$ meters, the area will be xy square meters, and the diagonal $\sqrt{x^2+y^2}$ meters by the Pythagorean theorem. Therefore $xy = 22$ and $x^2+y^2 = 200$.

    Now $(x+y)^2 = x^2+2xy+y^2 = x^2+y^2+2xy = 100+2\cdot 22 = 144$, and so $x+y = \sqrt{144} = 12$. Therefore the perimeter is $2\cdot 12 = 24$ meters, which is **A**.

12. Donald Duck can eat 2 pizzas in 3 minutes, while Goofy can eat 3 pizzas in 2 minutes. At these rates, how many pizzas can they eat together in an hour?

    (a) 54 **(b)** 96 **(c)** 130 **(d)** 216 **(e)** 250

    SOLUTION: Start by computing how many pizzas will each of them eat in 1 minute. That would be 2/3 for Donald Duck and 3/2 for Goofy. So each minute, both of them together will eat $2/3+3/2 = 13/6$ pizzas, which gives us 130 pizzas in 60 minutes. The answer is **C**.

13. If $m > 0$ and the points $(m, 3)$ and $(1, m)$ lie on a line with slope m, then $m =$

    (a) 1 **(b)** $\sqrt{2}$ **(c)** $\sqrt{3}$ **(d)** $\sqrt{5}$ **(e)** None of the above

    SOLUTION: The slope of the line through the points $(m.3)$ and $(1, m)$ is $\frac{m-3}{1-m}$. This must be equal to m, so $\frac{m-3}{1-m} = m$ which siplifies to $3 = m^2$. Since we know that $m > 0$, the solution is **C**, $\sqrt{3}$.

14. An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

    (a) 40% **(b)** 48% **(c)** 52% **(d)** 60% **(e)** None of the above

    SOLUTION: $100\% - 20\% = 80\%$ are coins, and out of the 80%, $100\% - 40\% = 60\%$ are gold. Therefore 60% **of** 80% are gold coins. That is $.6\cdot .8 = .48 = 48\%$. **B** is correct.

15. $\sqrt{\dfrac{8^{10}+4^{10}}{8^4+4^{11}}} =$

    (a) $\sqrt{2}$ **(b)** 16 **(c)** 32 **(d)** $12^{2/3}$ **(e)** None of the above

Solution: Write all the powers as powers of 2 using properties of exponents, and simplify:

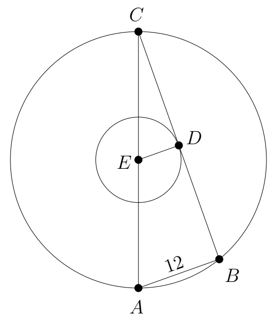
$$\sqrt{\frac{2^{30}+2^{20}}{2^{12}+2^{22}}} = \sqrt{\frac{2^{20}\left(1+2^{10}\right)}{2^{12}\left(1+2^{10}\right)}} = \sqrt{2^8} = 16$$

so **B** is correct.

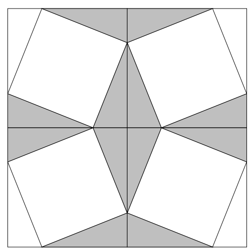
16. The ratio of the radii of two concentric circles (circles having the same center) is 1:3. If $\overline{AC}$ is a diameter of the larger circle, $\overline{BC}$ is a chord of the larger circle that is tangent to the smaller circle, and $AB = 12$, then the radius of the larger circle is

(a) 13 **(b)** 18 **(c)** 21 **(d)** 24 **(e)** None of the above

Solution: The angle ABC is a right angle. Because the chord $\overline{BC}$ is tangent to the small circle, the angle EDC is also a right angle. Therefore the two triangles, ABC and EDC, are similar. The sides $\overline{ED}$ and $\overline{EC}$ of the smaller triangle are the radii of the smaller circle and the larger circle, respectively. Therefore their ratio is 1:3. From similarity, the ratio of the corresponding sides in the larger triangle, $\overline{AB}$ and $\overline{AC}$, is also 1:3, and since $AB = 12$, we know that $AC = 36$. The radius of the larger circle is then half of the AC, which is 18. The answer is **B**.



17. The 12 in × 12 in configuration shown to the right made up of four tiles each represents a small square inscribed in a larger one. The shaded region has area 24 in^2. The side length of each of the inner squares is



(a) $2\sqrt{5}$ **(b)** $2\sqrt{3}$ **(c)** 3 **(d)** $\sqrt{28}$

(e) None of the above

Solution: We only need to look at one quarter of the configuration. The area of one of the four larger squares is $6 \times 6 = 36$ in^2. The shaded part is $24/4 = 6$ in^2. It consists of three identical triangles, each of area $6/3 = 2$ in^2. The area of the smaller square is the area of the larger square **minus** the area of the **four** identical triangle (three of them shaded, the fourth not shaded). So the area of the smaller square is $36 - 4 \cdot 2 = 28$ in^2. So the side of the smaller square is $\sqrt{28}$ in, that is **D**.

18. At a High School each freshman is taking exactly two of the following three courses: English, Math, Social Studies. The freshmen enrollments are: 20 in English, 17 in Math, 11 in Social Studies. How many freshmen are there?

(a) 24 **(b)** 25 **(c)** 26 **(d)** 27 **(e)** None of the above

Solution: Adding $20 + 17 + 11 = 48$ will count each of the freshmen twice (since each is enrolled in exactly two classes). So the answer is $48/2 = 24$, **A**.

19. Let a polynomial P be a degree four polynomial such that $0 = P(0) = P(1) = P(2) = P(-1)$ and $P(-2) = 12$. Then $P(3)$ equals

(a) $1/3$ **(b)** $-1/2$ **(c)** 1 **(d)** 2 **(e)** None of the above

Solution: A degree four polynomial with the given zeros will look like $P(x) = a \cdot x(x-1)(x-2)(x+1)$ for some constant a. Now $P(-2) = a(-2)(-3)(-4)(-1) = 24a$, and since this must be equal to 12, we know that $a = 1/2$. So $P(3) = \frac{1}{2}3(3-1)(3-2)(3+1) = \frac{1}{2}3 \cdot 2 \cdot 1 \cdot 4 = 12$. The answer is **E**, none of the above.

An alternative solution: Look at the polynomial $Q(x) = P(x + 1/2)$ instead. The fourth degree polynomial Q will have zeros -1.5, $-.5$, $.5$ and 1.5. Therefore it must be an **even** function. We also know that $Q(-2.5) = 12$, and we need to find $Q(2.5)$. But because Q is even, $Q(2.5) = Q(-2.5) = 12$.

20. Jody read a book in 3 days. During the first day she read 1/5 of the book, plus 16 pages. During the second day she read 3/10 of what remained,plus 20 pages. During the third day she read 3/4 of what remained, plus 30 pages. How many pages were there in the book?

(a) 225 **(b)** 240 **(c)** 265 **(d)** 270 **(e)** None of the above

Solution: First, let x be the number of pages left at the beginning of the last day. Then $\frac{3}{4}x + 30 = x$, and so $x = 120$. Now let y be the number of pages left at the beginning of the second day. Then $y - \frac{3}{10}y - 20 = 120$, or $\frac{7}{10}y = 140$, or $y = 200$. Finally, let z be the number of the pages in the book, then $z - \frac{1}{5}z - 16 = 200$, or $\frac{4}{5}z = 216$, or $z = 270$, so **D** is correct.

21. Determine a, b for which $(-4, -3)$ is a solution of the system

$$\begin{aligned} ax + by &= -26 \\ bx - ay &= 7 \end{aligned}$$

(a) $a = -2$, $b = 11$ **(b)** $a = 2$, $b = -6$ **(c)** $a = 9$, $b = 5$

(d) $a = 5$, $b = 2$ **(e)** None of the above

Solution: The numbers a and b must be solutions of the system

$$\begin{aligned} -4a - 3b &= -26 \\ -4b + 3a &= 7 \end{aligned}$$

Solving the system shows **D** is the answer.

22. The Jones family is remodeling their house. The width of a rectangular room is increased by 30% and the length of that room is increased by 25%. By what percentage is the area of the room increased?

(a) 55% **(b)** 62.5% **(c)** 7.5% **(d)** 75% **(e)** None of the above

SOLUTION: If the original width is x and the original length is y, then original area is xy and the new area is $1.3x \cdot 1.25y = 1.625xy$. So the area increased by 62.5%. The answer is **B**.

23. Which of these numbers is the smallest?

(a) $\dfrac{\sqrt{9^3}}{\sqrt[3]{27^{15}}}$ **(b)** $\sqrt{\sqrt[3]{3^{-60}}}$ **(c)** $\dfrac{1}{12}$ **(d)** $\dfrac{4^2 2^{-4}}{3^{11}}$ **(e)** $\dfrac{9^{-1}}{27 \cdot 3^5}$

SOLUTION: Write all numbers as powers of 2 or 3:

(a) $\dfrac{(3^2)^{3/2}}{(3^3)^{15/3}} = \dfrac{3^3}{3^{15}} = 3^{-12}$

(b) $\left(\left(3^{-60}\right)^{1/3}\right)^{1/2} = 3^{-10}$

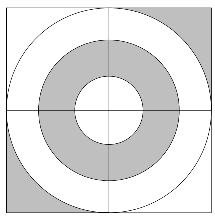
(c) $2^{-2}3^{-1}$

(d) $2^{4-4}3^{-11} = 3^{-11}$

(e) $\dfrac{3^{-2}}{3^{3+5}} = 3^{-2-8} = 3^{-10}$

Out of these, 3^{-12} is the smallest, so the answer is **A**.

24. The three circles have radii 1in, 2in and 3in. Which fraction of the square is shaded?



(a) $18 - \frac{3\pi}{2}$ **(b)** $\frac{3}{8}$ **(c)** $\frac{1}{2} - \frac{\pi}{24}$ **(d)** $\frac{5\pi}{6}$

(e) None of the above

SOLUTION: The area of the whole square is 36 in^2. The shaded area has two parts: the anulus between circles of radius 1 and 2, with area $4\pi - \pi = 3\pi$, and the two "corners", with area $\frac{36-9\pi}{2}$. So the total shaded area is

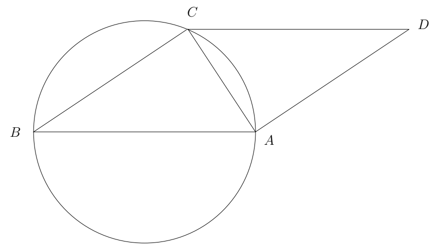
$$\frac{36 - 9\pi}{2} + 3\pi = 18 - \frac{9\pi}{2} + \frac{6\pi}{2} = 18 - \frac{3\pi}{2}$$

The fraction is then

$$\frac{18 - \frac{3\pi}{2}}{36} = \frac{1}{2} - \frac{\pi}{24}$$

so the answer is **C**.

25. A parallelogram ABCD is drawn so that the triangle ABC is an isosceles triangle inscribed inside circle of radius 12 and so that the one of its sides passes through the center. What is the area of the parallelogram?



(a) 288 **(b)** 36 **(c)** 72

(d) 144 **(e)** None of the above

SOLUTION: The triangle ABC is isosceles, so the point C actually lies on the perpendicular bisector of the segment $\overline{AB}$. Therefore the height of the triangle (and the parallelogram) is the radius, and so the area is base $\cdot$ height $= 24 \cdot 12 = 288$. The answer is **A**.