Saginaw Valley State University
2004 Math Olympics – Level II Solutions

These are examples of solutions to the problems on the Level II exam. There are certainly different ways to do each problem.

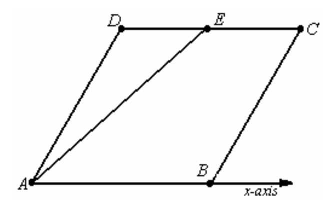
1) If $\log_2(32) - \log_3(27) = \log_4(\frac{x}{5})$, then x is

   a) 16 b) 20 c) 80 d) 125 e) None of the above

Solution: (C)

$\log_2(32) - \log_3(27) = \log_2 2^5 - \log_3 3^3 = 5 - 3 = 2$ since $\log_a a^x = x$. Therefore, we have $2 = \log_4\left(\frac{x}{5}\right)$. But $y = \log_a x \iff x = a^y$, so $\frac{x}{5} = 4^2$ or $x = 80$.

2) The rhombus $ABCD$ is made by gluing two equilateral triangles ABD and BCD along their common edge, and is placed so that AB is along the horizontal x-axis. The point E bisects the segment DC. Then the slope of the line AE is



   a) $\sqrt{3}$ b) $\frac{\sqrt{3}}{2}$ c) 1

   d) 1/2 e) None of the above

Solution: (B) Since Δ BCD is equilateral, $\overline{BE} \perp \overline{CD}$. Since ABCD is a rhombus, let x = AB = BC. In the right Δ BCE, $BE = x\sin 60^\circ = \frac{\sqrt{3}}{2}x$. So, the slope of the line

$$AE = \frac{\text{rise}}{\text{run}} = \frac{BE}{AB} = \frac{\sqrt{3}/2\, x}{x} = \frac{\sqrt{3}}{2}.$$

3) In a local election, 3000 votes were cast to elect one of the three candidates A, B, and C. Of the first 2000 votes, A received 45%, B received 35%, and the rest went to C. Of the remaining 1000 votes, 60% went to C, 25% went to B, and the rest went to A. Which of the following is true?

   a) A places first with 60% of the votes b) C wins the election

   c) A and C tie for first place d) C received 50 votes more than B

   e) None of the above

Solution: (D) Of the first 2000 votes, C received (100 – 45 – 35)% = 20%.
Of the remaining 1000 votes, A received (100 – 60 – 25)% = 15%.
The votes received by A = (0.45)(2000) + (0.15)(1000) = 1050
The votes received by B = (0.35)(2000) + (0.25)(1000) = 950
The votes received by C = (0.20)(2000) + (0.60)(1000) = 1000
Therefore C received 50 votes more than B.

4) Simplify $\dfrac{(x^2-x-6)(x+4)}{x^2+x-12}$

a) $x+3$ b) $\dfrac{x+3}{x+4}$ c) $\dfrac{x+2}{x+4}$ d) $x+2$ e) None of the above

Solution: (D) $\dfrac{(x^2-x-6)(x+4)}{x^2+x-12} = \dfrac{(x-3)(x+2)(x+4)}{(x+4)(x-3)} = x+2$

5) Let $f(x)=x^2+4x+2$. Let R be the quadrangle (quadrilateral) in the Cartesian plane whose vertices coincide with the x-intercepts of f and the vertices of the parabolas $y=f(x)$ and $y=-f(x)$. The area of R is

a) $4\sqrt{2}$ b) 12 c) 8 d) $2\sqrt{2}$ e) 2

Solution: (A)

The vertex of the parabola $f(x)=ax^2+bx+c$ is the point $(-b/2a, f(-b/2a))$, so the vertex of $f(x)=x^2+4x+2$ is D= (-2,-2). The vertex of $-f(x)$, the reflection of f across the x-axis, is B = (-2,2).

The quadratic formula gives the x – intercepts of f to be A = $(-2-\sqrt{2},0)$ and $C=(-2+\sqrt{2},0)$, so $AC=2\sqrt{2}$. The area of R is the area of ABCD = area of ΔABC + area of $\Delta CDA = 2(\text{area of }\Delta ABC) = 2(\tfrac{1}{2}(2\sqrt{2})(2)) = 4\sqrt{2}$.

6) In a class of 60 junior and senior students, 21 are girls, 34 are juniors, and 20 are senior boys. How many juniors are girls?

a) 15 b) 19 c) 6 d) 20 e) 8

Solution: (A) Construct a table with the bold, italicized numbers from the given information. We can then complete the table to find 15 junior girls.

	Girls	Boys	Total
Juniors	15	19	***34***
Seniors	6	***20***	***26***
Total	***21***	39	***60***

7) A large water tank has two drain outlets A and B. It takes 3 hours to drain the tank through A only, and 4 hours to drain the tank through B only. Suppose that we started draining the full tank using both outlets, but after a while outlet B was closed letting the rest of the water drain through A. Given that it took a total of $2\frac{1}{4}$ hours to drain the tank, how long did it take before closing outlet B?

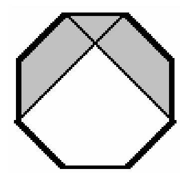
a) $1\frac{1}{4}$ hrs b) 1 hr c) $\frac{7}{12}$ hr d) $\frac{5}{12}$ hr e) None of the above

Solution: (B) Drain A is open $2^1/_4$ hours. Let t be the amount of time drain B is open. The rate for A is $^1/_3$ of a tank per hour; and the rate for B is $^1/_4$. So $^1/_3 \cdot {}^9/_4 + {}^1/_4 t = 1$. $^1/_4 t = {}^1/_4$, So $t = 1$.

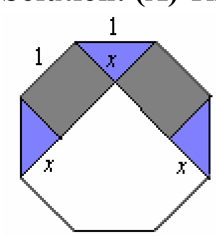
8) The figure shows a regular octagon of side 1. The area of the shaded region is

a) $\sqrt{2} + \frac{3}{4}$ b) $3\sqrt{2}$ c) $\frac{3\sqrt{2}}{4}$

d) $1 + \sqrt{2}$ e) None of the above



Solution: (A) The area of the shaded region is 2 × the area of the rectangle plus 3 × the area of the right triangle. Since the triangles are isosceles right triangles (vertex angle is 135°) with hypotenuse 1, $x = \frac{1}{\sqrt{2}}$. The area of the rectangle is $1 \times x$, the area of the triangle is $\frac{1}{2} x^2$. So the area of the shaded region is $2x + {}^3/_2 x^2$.. So the total area is

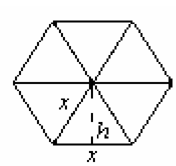


$$2 \cdot \frac{1}{\sqrt{2}} + \frac{3}{2} \cdot \frac{1}{2} = \sqrt{2} + \frac{3}{4}.$$

9) The length of the side of a regular hexagon with area 1 in^2 is

a) $\sqrt{\frac{3}{4}}$ b) $\frac{3}{4}$ c) $\frac{3\sqrt{2}}{2}$ d) $\sqrt[4]{\frac{3}{4}}$ e) None of the above

Solution: (E) Let x be the length of the sides. A regular hexagon consists of 6 equilateral triangles of side length x as shown. The area of each triangle is $\frac{1}{2} xh$, where h is the height of the triangle. $(\frac{1}{2} x)^2 + h^2 = x^2$. So $h = \frac{\sqrt{3}}{2} x$.



The area of each triangle, then, is $\frac{\sqrt{3}}{4} x^2$. The area of the hexagon is $6\left(\frac{\sqrt{3}}{4} x^2\right)$, which is equal to 1. Solving for x gives $x = \sqrt[4]{\frac{4}{27}}$.

10) If all the roots (zeros) of the polynomial $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx - 420$ are integers larger than 1, then $f(4)$ equals

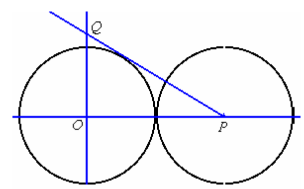
a) 0 b) -6 c) 12 d) -12 e) None of the above

Solution: (C) f must have 5 (not necessarily distinct) roots $d_1, d_2, \ldots d_5$. f factors as $(x - d_1)(x - d_2)(x - d_3)(x - d_4)(x - d_5)$. The product $d_1 \cdot d_2 \cdot d_3 \cdot d_4 \cdot d_5$ must be equal to 420, which factors as $2^2 \cdot 3 \cdot 5 \cdot 7$. All of the roots are integers larger than 1, so they must be 2, 2, 3, 5, and 7. So $f(x) = (x-2)^2(x-3)(x-5)(x-7)$. Putting in $x = 4$ gives 12.

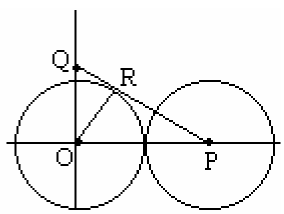
11) In the figure, the two circles both have radius 1 and centers O and P respectively. The line PQ is tangent to the first circle and passes through the center of the second, and the line OQ is perpendicular to the line OP joining the centers. The length of the segment OQ is

a) $\sqrt{3}$ b) $\frac{\sqrt{3}}{2}$ c) $\frac{\sqrt{2}}{3}$

d) $\frac{2\sqrt{3}}{3}$ e) None of the above



Solution: (D)



Let R be the point of tangency for the line PQ and the first circle. The line OR has length 1 and is perpendicular to the line PQ. OP has length 2 so the angle ∠OPR has a sine of ½, so it is 30°. This means the length of OQ is 2tan30°.

12) If $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$ for all $x \neq 0,1$ and $0 < \theta < \pi/2$ then $f(\sec^2\theta) =$

a) $\sin^2\theta$ b) $\cos^2\theta$ c) $\tan^2\theta$ d) $\csc^2\theta$ e) None of the above

Solution: (A) If $\frac{x}{x-1} = \sec^2\theta$, then $\cos^2\theta = \frac{x-1}{x} = 1 - \frac{1}{x}$. Therefore,

$\frac{1}{x} = 1 - \cos^2\theta = \sin^2\theta$

13) Suppose that x satisfies the equation $\sin x = \frac{1}{\tan x}$. Then $\cos x$ equals

a) 0 b) $\frac{\sqrt{3}}{2}$ c) $\frac{\sqrt{5}-1}{2}$ d) $\frac{\sqrt{5}}{4}$ e) None of the above

Solution: (C) Since $\sin x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$, $\sin^2 x = \cos x$, $1 - \cos^2 x = \cos x$, or

$\cos^2 x + \cos x - 1 = 0$, or $\cos x = \frac{-1+\sqrt{5}}{2}$ by the Pythagorean Theorem. (The other solution is not possible since $\frac{-1-\sqrt{5}}{2} < -1$, i.e. not in the range of $\cos x$.)

14) The domain for the inverse of the function $f(x) = \sqrt{x-5}$ is:

a) $(5, \infty)$ b) The same as the domain of f. c) $[0, \infty)$

d) both a and b are true e) $x = 5$

Solution: (C) The domain of the inverse of f is equal to the range of $f(x) = \sqrt{x-5}$, which is $[0, \infty)$.

15) A cashier found that he was often asked to give change for a dollar to people who made no purchase but wanted a dime or two nickels for a parking meter. He started thinking one day about the number of ways he could make change using the coins half-dollar, quarter, dime, nickel, and penny. If he gave no more than four of any coin, in how many different ways could he give change for a dollar to people who needed a dime or two nickels for parking?

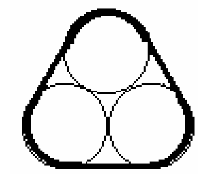
a) 6 b) 8 c) 10 d) 12 e) None of the above

Solution: (B) The possibilities are

Dime	1	1	2	2	3	3	4	4
Nickel	3	3	1	1	4	4	2	2
Quarter	3	1	1	3	2	0	2	0
Half dollar	0	1	1	0	0	1	0	1

The total number of possibilities is 8.

16) Three cylindrical drums of 2-foot diameters are to be securely fastened in the form of a triangle by a steel band. What length of band will be required?



a) $6+2\pi$ ft b) 12π ft c) $12 + 2\pi$ ft

d) $6 + 3\pi$ ft e) None of the above

Solution: (A) The part of the steel band touching each drum is 1/3 of the circumference of each circle, that gives a total length of $3\times(1/3\cdot 2\pi r) = 2\pi$ since r = 1. The remaining three parts of the band between two drums equals 2r = 2, that gives a total of 6. Thus the length of band is $6 + 2\pi$.

17) A high school runner wants to run a 5-minute mile in his next race. That's an average of 12 miles per hour. He figures he'll start out slowly and finish strong. He runs the first half-mile at 10 miles per hour. How fast must he run the last half-mile so that his time for the race is 5 minutes?

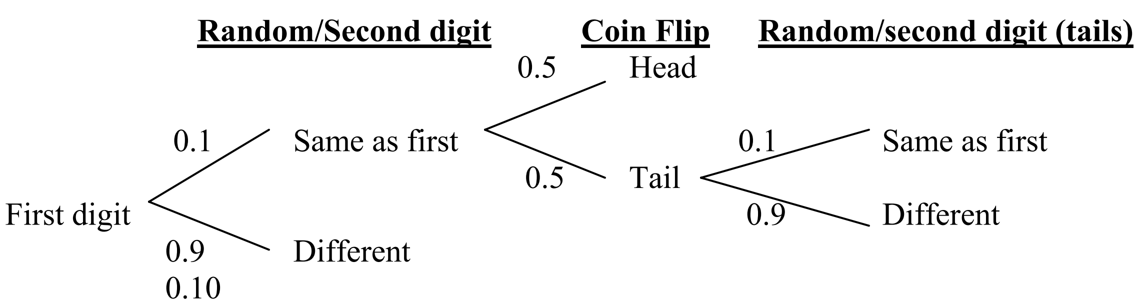
a) 11 mi/hr b) 13 mi/hr c) 14 mi/hr d) 15 mi/hr e) None of the above

Solution: (D) The first half-mile at 10 mph = $1/6$ mile per minute. It took the runner 3 minutes to finish the first half-mile. He needs to finish the second half-mile in 2 minutes to have a 5 minute mile, which is 0.25 mile per minute = $0.25 \times 60 = 15$ mph.

18) A pair of digits is produced as follows. The first digit is randomly selected and registered. A second digit is randomly selected and registered if it differed from the first one. Otherwise a fair coin is flipped. If the coin is heads the digit is registered, and if the coin is tails, a digit is randomly selected and registered. What is the probability that the two registered digits are equal?

a) 0.1 b) 0.055 c) 0.05 d) 0.045 e) None of the above

Solution (B): After the first digit is selected, consider the following tree of probabilities according to the event at the top (in sequence from left to right).



The two ways to get two equal digits is if the second digit was randomly selected the same and a head came up on the coin flip ($0.1\times 0.5 = 0.05$) OR if the second digit was randomly selected the same, a tail was flipped, then the same digit was randomly selected again ($0.1\times 0.5\times 0.1 = 0.005$). Therefore, the probability is $0.05 + 0.005 = 0.055$.

19) Let $y = mx + b$ be the image when the line $x - 3y + 11 = 0$ is reflected across the x-axis. The value of $m + b$ is

a) -6 b) -5 c) -4 d) -3 e) None of the above

Solution: (C) Since the point (x, y) maps to $(x, -y)$ under a reflection across the x-axis, the image of the line $x - 3y + 11 = 0$ is $x - 3(-y) + 11 = x + 3y + 11 = 0$. Solving for y, we have $y = -\frac{1}{3}x - \frac{11}{3}$, so $m + b = -\frac{1}{3} - \frac{11}{3} = -4$.

20) Let the operation $*$ be defined as $a * b = b + \frac{1}{a}$. The value of $(1 * 2) * 4$ is

a) $3\frac{1}{4}$ b) $2\frac{1}{4}$ c) $4\frac{1}{3}$

d) $1\frac{3}{4}$ e) None of the above

Solution: (C) $1 * 2 = 2 + \frac{1}{1} = 3$, so $(1 * 2) * 4 = 3 * 4 = 4 + \frac{1}{3}$.

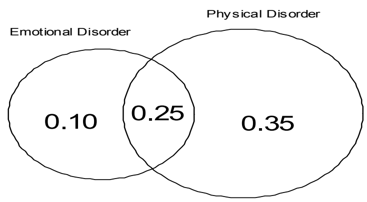
21) One hundred students at Century High School took an exam last year, and their mean score was 100. The number of non-seniors taking the exam was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of the non-seniors. What was the mean score of the seniors?

a) 100 b) 112.5 c) 120 d) 125 e) None of the above

Solution: (D) Let S = the number of seniors and N = the number of non-seniors. Since N is 50% more than S, we have N = S + 0.5 S=1.5 S. Now S + N = 100, so S + 1.5 S = 2.5 S = 100, which gives S = 40 and, therefore, N = 60.
Let $\sum_A, \sum_S, \sum_N$ be the sum of the scores of All the students, the Seniors, and the Non-seniors, respectively, and $\overline{x}_S, \overline{x}_N$ the average score of Seniors and Non-seniors, respectively. From the formula for the average, $\sum_S = 40\overline{x}_S$, $\sum_N = 60\overline{x}_N$, and $\sum_A = 100 \cdot 100 = 10,000$ (given values). The mean score for the seniors was 50% higher than the non-seniors, so $\overline{x}_S = 1.5\overline{x}_N$ or $\overline{x}_N = \frac{2}{3}\overline{x}_S$. Substituting, $\sum_N = 60 \cdot \frac{2}{3}\overline{x}_S = 40\overline{x}_S$.
Now, $10,000 = \sum_S + \sum_N = 40\overline{x}_S + 40\overline{x}_S = 80\overline{x}_S$, so $\overline{x}_S = 125$.

22) A survey indicated that in a particular city, 60% of the patients visiting a doctor have a physical disorder, 35% have an emotional disorder, and 25% have both a physical disorder and an emotional disorder. What is the probability of a person having neither a physical nor an emotional disorder visiting a doctor?

a) 0.35 b) 0.30 c) 0.25 d) 0.70 e) None of the above



Solution: (B) Constructing the Venn Diagram, since 25% have both and 60% have a physical disorder, 35% have a physical disorder but not an emotional disorder. Similarly, 35% - 25% = 10% have an emotional disorder but no physical disorder. This accounts for 70% that have a physical disorder or an emotional disorder (or both), and, so 30% have neither a physical nor an emotional disorder.

23) For what values of k will the remainder of division of $x^2 + kx + 4$ by $x-1$ be twice the remainder of division of $x^2 + kx + 4$ by $x+1$?

a) 5/3 b) 4 c) ¼ d) 2/3 e) None of the above

Solution (A): The remainder of a polynomial $P(x)$ after division by $x-r$ is $P(r)$ by the Remainder Theorem, so if $P(x) = x^2 + kx + 4$, then division by $x-1$ results in a remainder of $P(1) = 1+k+4 = 5+k$. Division by $x+1$ has a remainder of $P(-1) = 1-k+4 = 5-k$. We want to know k when $5+k = 2(5-k)$, which is $k = \frac{5}{3}$.

Note: The remainders can also be obtained by actually doing the division (without The Remainder Theorem).

24) The solution of the equation $\log(\log x) = 2$ is

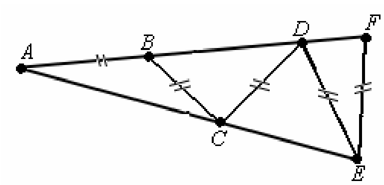
a) 10^3 b) 10 c) 10^{10} d) 10^{100} e) None of the above

Solution: (D) Since $\log a = b$ implies $a = 10^b$, $\log(\log x) = 2$ implies $\log x = 10^2 = 100$. Therefore, $x = 10^{100}$.

25) Given the isosceles triangle *AEF* (where *AE=AF*) with a path of 5 congruent segments A – B – C – D – E – F, the degree measure of angle A is

a) 10° b) 15° c) 20°

d) 30° e) None of the above



Solution:(C) Note that $\angle DBC + \angle ABC = 180 = \angle A + \angle ACB + \angle ABC$, so $\angle DBC = \angle A + \angle ACB$. This is the exterior angle theorem: The measure of an exterior angle to a triangle is equal to the sum of the two non-adjacent interior angles. Since base angles of the isosceles triangle ABC are congruent $\angle A = \angle ACB$. If we let $a = \angle A = \angle ACB$, then $\angle DBC = \angle BDC = 2a$ and exterior angle $\angle DCE$ to ΔADC is $a + 2a = 3a = \angle DEC$. Exterior angle $\angle FDE$ to ΔADE, then, is $3a + a = 4a = \angle F$. Since ΔAEF is isosceles, $\angle FEA = 4a$ as well. The three angles in triangle AEF sum to $a + 4a + 4a = 180$. This gives $a = 20$.